SOLUTIONS

1.

a) Rotate about the x-axis, use the slicing method

$$V = \int_0^1 \pi \left[4 - \frac{1}{(1 + x)^2}\right] dx = \pi \left[4x + \frac{1}{1 + x}\right]_0^1 = \frac{7\pi}{2}$$

b) Rotate about the y-axis, use the cylindrical shell method

$$V = 2\pi \int_0^1 x \left(2 - \frac{1}{1 + x}\right) dx$$

$$= 2\pi \left[ x^2 \bigg|_0^1 - \int_0^1 \frac{x}{1 + x} dx \right]$$

$$= 2\pi \left[ 1 - \int_0^1 \left(1 - \frac{1}{1 + x}\right) dx \right]$$

$$= 2\pi \left(1 - (x - \ln(1 + x)) \bigg|_0^1\right)$$

$$= 2\pi \ln 2$$

2. a) For \(x \neq y\), we have

$$f(x, y) = \frac{x^4 - y^4}{x^2 - y^2}$$

The latter expression has the value \(2x^2\) at points of the line \(x = y\). Therefore, we extend the definition of \(f(x, y)\) so that \(f(x, x) = 2x^2\) then the resulting function will be equal to \(f(x, y) = x^2 + y^2\) everywhere, and continuous everywhere

b) \(\lim_{h \to 0} \frac{f(h, 0) - 0}{h} = \lim_{h \to 0} \frac{\sin h^3}{h^3} = \lim_{h \to 0} \frac{\cos h^3 (3h^2)}{3h^2} = 1\)

\(\lim_{k \to 0} \frac{f(0, k) - 0}{k} = \lim_{k \to 0} \frac{\sin k^3}{k^3} = \lim_{k \to 0} \frac{\cos k^3 (3k^2)}{3k^2} = 1\)
3. \( f(x, y) = \ln(x^3 + y^3) \)

- \( f_1(x, y) = \frac{3x^2}{x^3 + y^3}, \quad f_1(1, 2) = \frac{1}{3} \)
- \( f_2(x, y) = \frac{3y^2}{x^3 + y^3}, \quad f_2(1, 2) = \frac{4}{3} \)

a) \( \nabla f(1, 2) = \frac{1}{3}i + \frac{4}{3}j \)

b) \( f(1, 2) = \ln 9 \), the point of tangency is \((1, 2, \ln 9)\). Equation of the tangent plane:
   \[ z = \ln 9 + \frac{1}{3}(x - 1) + \frac{4}{3}(y - 2) \]

c) \( \frac{1}{3}(x - 1) + \frac{4}{3}(y - 2) = 0 \quad \Rightarrow \quad x + 4y = 9 \)

d) Equation of the normal line:
   \[ \frac{x - 1}{1/3} = \frac{y - 2}{4/3} = \frac{z - \ln 9}{-1} \]

4. a) The point \((x, y, z)\) must be a critical function Lagrangian function
   \[ L = x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 3). \]

To find these critical points we have

\[
\begin{align*}
\frac{\partial L}{\partial x} &= 2x + \lambda = 0 \\
\frac{\partial L}{\partial y} &= 2y + 2\lambda = 0 \\
\frac{\partial L}{\partial z} &= 2z + 2\lambda = 0 \\
\frac{\partial L}{\partial \lambda} &= x + 2y + 2z - 3 = 0.
\end{align*}
\]

The first three equations yields \( y = z = -\lambda, \quad x = -\lambda/2 \). Substituting these into the fourth equation we get \( \lambda = -2/3 \), so that the critical point is \( (\frac{1}{3}, \frac{2}{3}, \frac{2}{3}) \), whose distance from the origin is 1.
b) $f(x, y) = xy e^{-x+y}$

$f_1(x, y) = y(1-x)e^{-x+y}$

$f_2(x, y) = x(1+y)e^{-x+y}$

$A = f_{11}(x, y) = (-2y + xy)e^{-x+y}$

$B = f_{12}(x, y) = (1-x+y-xy)e^{-x+y}$

$C = f_{22}(x, y) = (2x + xy)e^{-x+y}$

Critical points are $(0, 0)$ and $(1, -1)$.

At $(0, 0)$: $A = 0$, $B = 1$ and $C = 0$, so it is a saddle point.

At $(1, -1)$: $A = e^{-2}$, $B = 0$ and $C = e^{-2}$, so it is a local minimum point.