1. (a) Starting with the power series representation

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n
\]

determine the Taylor series representation of \( f(x) = \frac{1}{x^2} \) in powers of \( x - 5 \).

**Solution:**

First, differentiate the function \( \frac{1}{1 - x} \) and its series representation with respect to \( x \) to get

\[
\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{n=1}^{\infty} n x^{n-1}.
\]

Let \( t = x - 5 \), i.e., \( x = 5 + t \). So, we have

\[
f(x) = \frac{1}{x^2} = \frac{1}{(5 + t)^2} = \frac{1}{25} \left(\frac{1}{1 + \frac{t}{5}}\right)^2.
\]

Substitute \( \frac{-t}{5} \) for \( x \) in \( \frac{1}{(1 - x)^2} = \sum_{n=1}^{\infty} n x^{n-1} \), to get

\[
\frac{1}{\left(1 + \frac{t}{5}\right)^2} = \sum_{n=1}^{\infty} n \left(\frac{-t}{5}\right)^{n-1}.
\]

Then multiply the resulting equation by \( \frac{1}{25} \) to arrive at

\[
\frac{1}{25} \sum_{n=1}^{\infty} n \left(\frac{-t}{5}\right)^{n-1}.
\]

Finally, use the transformation \( t = x - 5 \) to find

\[
\frac{1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{5^n} (x - 5)^{n-1}.
\]

(b) If \( S(x) = \int_{0}^{x} \sin \left(\frac{t^2}{2}\right) dt \), find \( \lim_{x \to 0} \frac{x^3 - 3S(x)}{x^7} \)

**Solution:**

Maclaurin series for \( \sin \left(\frac{t^2}{2}\right) = t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \ldots \)

\[
S(x) = \int_{0}^{x} \sin \left(\frac{t^2}{2}\right) dt = \int_{0}^{x} \left(\frac{t^2}{2} - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \ldots\right) dt
\]

\[
S(x) = \left(\frac{t^3}{3} - \frac{t^7}{7 \times 3!} + \frac{t^{11}}{11 \times 5!} - \ldots\right)|_{0}^{x}
\]

\[
S(x) = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - \ldots
\]

Substitute \( S(x) \) in the limit:
\[
\lim_{x \to 0} \frac{x^3 - 3S(x)}{x^7} = \lim_{x \to 0} \frac{x^3 - 3 \left( \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - \cdots \right)}{x^7}
\]
\[
= \lim_{x \to 0} \left( \frac{3}{7 \times 3!} - \frac{3 x^4}{11 \times 5!} + \cdots \right) = \frac{3}{7 \times 3!} = \frac{1}{14}
\]

2. Find the Fourier series of the function \(f(t)\) with period 2 whose values in the interval \([-1, 1]\) are given by
\[
f(t) = \begin{cases} 
0 & \text{if } 1 \leq t < 0 \\
t & \text{if } 0 \leq t < 1
\end{cases}
\]

**Solution:**
The Fourier coefficients of \(f\) are as follows:
\[
a_0 = \frac{1}{2} \int_{-1}^1 f(t) \, dt = \frac{1}{2} \int_0^1 t \, dt = \frac{1}{4},
\]
\[
a_n = \int_{-1}^1 f(t) \cos(n \pi t) \, dt
\]
\[
= \int_0^1 t \cos(n \pi t) \, dt
\]
\[
= \frac{(-1)^n - 1}{n^2 \pi^2}
\]
\[
= \begin{cases} 
-2/(n \pi)^2 & \text{if } n \text{ is odd} \\
0 & \text{if } n \text{ is even}
\end{cases}
\]
and
\[
b_n = \int_0^1 t \sin(n \pi t) \, dt = \frac{(-1)^n}{n \pi}.
\]
Hence, the Fourier series of \(f\) is
\[
\frac{1}{4} - \frac{2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos((2k-1)\pi t) - \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \sin(k\pi t).
\]

3. (a) Find the general solution to \(x \frac{dy}{dx} + 3y = 6x^3\).

**Solution:**
\[
\frac{3}{x} y = 6x^2.
\]
\[
\mu(x) = e^{\int \frac{3}{x} \, dx} = x^3.
\]
Multiply both sides by \(\mu(x)\) to obtain
\[
x^3 \frac{dy}{dx} + 3x^2 y = 6x^5.
\]
That is, \((x^3 y)' = 6x^5\). Integrate both sides to get \(y = x^3 + Cx^{-3}\). 

(b) Solve the integral equation 
\[ y(x) = 2 + \int_0^x e^{-y(t)} \, dt, \]
where \( y(0) = 2 \).

**Solution:**
\[
\frac{dy}{dx} = e^{-y}
\]
\[ e^y \, dy = dx \]
\[ e^y = x + C \Rightarrow y = \ln(x + C) \]
\[ y(0) = \ln(C) = 2 \Rightarrow C = e^2 \]
\[ y = \ln(x + e^2) \]

4. (a) Find the general solution of the following ordinary differential equation
\[ e^x(y - x) \, dx + (1 + e^x) \, dy = 0. \]

**Solution:**
Let \( M = e^x(y - x) \) and \( N = (1 + e^x) \). Since
\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^x,
\]
the given equation is exact and there exists a solution \( \phi \) s.t
\[
\frac{\partial \phi}{\partial x} = M, \quad \frac{\partial \phi}{\partial y} = N.
\]
Then the solution is of the form (by integrating \( N \) with respect to \( y \))
\[
\phi(x, y) = y + ye^x + g(x), \quad (1)
\]
where \( g \) is an arbitrary function of \( x \). Differentiation of (1) with respect to \( x \) and equating resulting equation to \( N \) we obtain
\[
g'(x) = -xe^x,
\]
which implies that
\[
g(x) = -xe^x + e^x + c.
\]
Hence, the solution is obtained as
\[
\phi(x, y) = y + ye^x - xe^x + e^x + c = 0.
\]

(b) Find the general solution of the following homogeneous differential equation
\[
x \frac{dy}{dx} = y - x \sin \left( \frac{y}{x} \right).
\]
Solution:
The given differential equation is of the type homogeneous and use the substitution $v = \frac{y}{x}$ for

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2\left(\frac{y}{x}\right),$$

which implies the equation as

$$v + x \frac{dv}{dx} = v - \sin^2(v).$$

Thus

$$-\frac{dv}{\sin^2(v)} = \frac{dx}{x},$$

which yields the solution as

$$\cot\left(\frac{y}{x}\right) = \ln(xc).$$