Expected values of the number of failures for two populations under joint Type-II progressive censoring

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The model in which joint Type-II progressive censoring is implemented on two samples from different populations in a combined manner is considered and the probabilities of failures are discussed. This model may have relevance for practical applications, when an experimenter need to know the expected values of the number of failures for each population. This knowledge plays an important role in the selection of appropriate sampling plans and the construction of efficient estimators for parameters. The formula allowing numerical computation of the expected value of the number of failures for each of the two populations is given. Also, a detailed numerical study of this expected value is carried out for different parametric families of distributions.

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1. Introduction

Recently, Balakrishnan and Rasouli (2008) have introduced a joint censoring scheme which is important in performing comparative life-tests of two identical products manufactured in different units under the same conditions. To describe this scheme, suppose products are being manufactured by two lines under the same conditions and two samples of products of sizes \( m \) and \( n \) are selected from these two lines and are placed simultaneously on a life-testing experiment. Then, based on cost considerations and time restrictions for completion of the test, suppose the researcher chooses to terminate the life-testing experiment as soon as a certain number of failures occur. In this situation, either point or interval estimation of the mean lifetimes of products manufactured by two lines may be of interest. Under joint Type-II censoring, specimens of two products under study are placed on a life-test simultaneously, the successive failure times and the corresponding product types are recorded, and the experiment is terminated as soon as a specified total number of failures (say, \( k \)) occurs.

Suppose \( X_1, \ldots, X_m \), the lifetimes of \( m \) specimens of product \( A \), are independent and identically distributed (i.i.d.) random variables, with cumulative distribution function (c.d.f.) \( F(x) \) and probability density function (p.d.f.) \( f(x) \). Let \( Y_1, \ldots, Y_n \) be i.i.d. random variables with c.d.f. \( G(x) \) and p.d.f. \( g(x) \), representing the lifetime of \( n \) specimens of product \( B \). Further, suppose \( W_1 \leq \cdots \leq W_N \) denote the order statistics of the \( N = m + n \) random variables \( \{X_1, \ldots, X_m; Y_1, \ldots, Y_n\} \). Then, under the joint Type-II censoring scheme, the observable data consists of \( (Z, W) \), where \( W = (W_1, \ldots, W_k) \), with \( k (1 \leq k < N) \) being a pre-fixed integer, and \( Z = (Z_1, \ldots, Z_k) \) with \( Z_i = 1 \) or \( 0 \) accordingly as \( W_i \) is from an \( X \)- or \( Y \)-failure.

A joint censoring scheme has been previously considered in the literature. For example, Basu (1968), discussed a generalized Savage statistic, while Johnson and Mehrotra (1972) studied the locally most powerful rank tests. Bhattacharyya and Mehrotra (1981) considered the problem of testing the equality of two distributions, under the assumption of exponentiality. The developments under this sampling scheme have been reviewed by Bhattacharyya (1995) in Chapter 7.
of Balakrishnan and Basu (1995) and all research has focused on nonparametric and parametric tests of hypotheses. Balakrishnan and Rasouli (2008) have studied the exact likelihood inference for two exponential populations under joint Type-II censoring.

If a researcher desires to remove live units at points other than the termination point of the life-test, the above described scheme will be of no value because the joint Type-II censoring does not allow for units to be lost or removed from the test at points other than the final termination point. More general censoring schemes are therefore required.

Rasouli and Balakrishnan (in press) consider the joint Type-II Progressive Censoring (JPC) scheme. Under the JPC scheme, m units of product A with c.d.f. \( F(x) \) and p.d.f. \( f(x) \) and n units of product B with c.d.f. \( G(y) \) and p.d.f. \( g(y) \) are placed on test at time zero (thus, we have \( N = m + n \) units on test), and k failures from both product A and product B are going to be observed. Immediately following the first failure, \( r_1 \) of the surviving units from both of product A and product B are randomly selected and removed (we can partition \( r_1 \) into \( r_1^A \) and \( r_1^B \) such that \( r_1^A \) and \( r_1^B \) are the number of failures of product A and the number of failures of product B respectively \((r_1 = r_1^A + r_1^B)\)). Then, immediately following the second observed failure, \( r_2 \) of the surviving units from both product A and product B are removed \((r_2 = r_2^A + r_2^B)\), and so on. This experiment terminates at the time when the \( k \)th failure is observed and the remaining \( r_k = N - r_1 - r_2 - \cdots - r_{k-1} \) surviving units from two populations are all removed. If \( r_1 = r_2 = \cdots = r_{k-1} = 0 \), then \( r_k = N - k \), which corresponds to the joint Type-II censoring. If \( r_1 = r_2 = \cdots = r_k = 0 \), then \( N = k \), which corresponds to the complete sample.

Several authors have addressed inferential issues based on progressive Type-II censored samples (for example, see Gibbons and Vance (1983), Balakrishnan et al. (1987), Balakrishnan and Sandhu (1995) and Bairamov and Eryilmaz (2006)). Viveros and Balakrishnan (1994) discussed interval estimation of life parameters based on progressive Type-II censored data. Kuy and Kaya (2007) examined a simultaneous confidence interval for parameters of Pareto distribution based on progressive Type-II censored data. (For more details see also Balakrishnan and Aggarwala (2000) and Balakrishnan (2007).) Rasouli and Balakrishnan (in press) developed exact inferential methods based on MLEs and compared their performance with those based on approximate, Bayesian and bootstrap methods, based on the JPC scheme under the assumption of exponentiality for both samples. For recent works on progressive censoring see also Balakrishnan and Rasouli (2008), Huang and Wu (2008), Wu (2008), Sanjel and Balakrishnan (2008), Soliman (2008) and Balakrishnan et al. (2008).

In practical applications, a researcher may need to know the expected values of the number of failures for each of the two populations. This information is important for the selection of an appropriate sampling plan, because in a statistical inference for parameters, the number of failures for a population are directly related to the estimator efficiency. In this study, we consider a joint Type-II progressive censoring scheme for two populations when the censoring is implemented on the two samples in a combined manner. Section 2 presents the details of the proposed model.

The structure of probability of failures is discussed in Section 3. The formula which is most valid for the computation of expected values of the number of failures for each of the two populations is given in Section 4. Finally, a detailed numerical study of the expected values are carried out for various classes of life distributions such as Gamma, Pareto and Weibull. It is important to examine how well the proposed method works for approximation of the expected values (A.E.) of the number of failures. To study the performance of this approach 1,000, JPC samples from different distributions and censoring schemes are simulated and the simulated expected values of the number of failures are obtained. The simulation results are also shown in Tables 1–3 in the Appendix. It can be seen that the approximate expected values of the number of failures are close to the simulated expected values of the number of failures for different distributions and censoring schemes. Hence, the proposed method is reliable for approximating the expected values of the number of failures.

2. The model

Suppose \( m \) and \( n \) independent units are placed in a test with the corresponding lifetimes being identically distributed with p.d.f.'s \( f(x; \theta_1) \) and \( g(y; \theta_2) \) and c.d.f.'s \( F(x; \theta_1) \) and \( G(y; \theta_2) \), \( \theta_1 = (\theta_{11}, \theta_{12}, \ldots, \theta_{1d_1}), \theta_2 = (\theta_{21}, \theta_{22}, \ldots, \theta_{2d_2}), \theta_1 \in \mathbb{R}^{d_1}, \theta_2 \in \mathbb{R}^{d_2} \), respectively. Under the JPC with censoring scheme \( R = (R_1 = r_1, \ldots, R_k = r_k) \), the observable data consists of \((\mathbf{W}^R, \mathbf{Z}, \mathbf{R})\) where \( \mathbf{W}^R = (W_{1:k,N}^R, W_{2:k:N}^R, \ldots, W_{k:k,N}^R) \) and \( \mathbf{Z} = (Z_1, \ldots, Z_k) \) with \( Z_i = 1 \) if \( W_i \) is from \( X \) (product A) failure and \( Z_i = 0 \) if \( W_i \) is from \( Y \) (product B) failure and \( \mathbf{R} = (r_1 = r_1', \ldots, r_{k-1} = r_{k-1}') \).

Let \( M_k = \sum_{i=1}^k Z_i \) denote the number of \( X \)-failures in \( \mathbf{W}^R \) and \( N_k = k - M_k \) (i.e., the number of \( Y \)-failures in \( \mathbf{W}^R \)). The likelihood function of \((\mathbf{W}^R, \mathbf{Z}, \mathbf{R})\) is derived by Rasouli and Balakrishnan (in press) as:

\[
L(\theta_1, \theta_2; \mathbf{W}^R, \mathbf{Z}, \mathbf{R}) = C \prod_{i=1}^{k} \left[ (f(w_i; \theta_1))^{z_i} [g(w_i; \theta_2)]^{1-z_i} \right] \left[ F(w_i; \theta_1) \right]^{z_i} \left[ G(w_i; \theta_2) \right]^{1-z_i},
\]

where \( w_1 < w_2 < \cdots < w_k \), and \( m_0 \equiv n_0 \equiv 0 \) and \( \bar{F} = 1 - F \) and \( \bar{G} = 1 - G \) are the survival functions of the two populations. Note that \( r_k = N - k - r_1 - r_2 - \cdots - r_{k-1} \), \( r_i = m - m_i - r_i - r_i' - \cdots - r_{i-1}' \). For \( i = 1, 2, \ldots, k \), \( r_i' = r_i - r_i' \) and \( C = D_1 D_2 \) with

\[
D_1 = \prod_{j=1}^{k} \left[ \left( m - \sum_{i=1}^{j-1} z_i - \sum_{i=1}^{j-1} r_i' \right) z_j + \left( n - \sum_{i=1}^{j-1} (1 - z_i) - \sum_{i=1}^{j-1} r_i' \right) (1 - z_j) \right],
\]
\[
D_2 = \prod_{j=1}^{k-1} \left\{ \left( \frac{m-n-j-\sum_{i=1}^{j-1} r_i}{r_j} \right) \left( \frac{n-j-\sum_{i=1}^{j-1} r_i}{n-j-\sum_{i=1}^{j-1} r_i'} \right) \right\}.
\]

From (1), we have the joint density of \((W^R, Z, R')\) as

\[
f_{W^R, Z, R'}(w, z; \theta_1, \theta_2) = C \prod_{i=1}^{k} \left[ \left( f(w_i; \theta_1) \right)^{z_i} \left( g(w_i; \theta_2) \right)^{1-z_i} \right] \left[ \tilde{f}(w_i; \theta_1) \right]^{r_i'} \left( \tilde{g}(w_i; \theta_2) \right)^{r_i''}
\]

\[0 < w_1 < w_2 < \ldots < w_k < \infty.\]

In the following theorem, the expressions for the probability of failures are given.

**Theorem 3.1.** The joint density of \((W^R, Z, R')\) is

\[
f_{W^R, Z, R'}(w, z; \theta_1, \theta_2) = f_{W^R, Z=R'=r'}(w; \theta_1, \theta_2 | Z = z, R' = r') P(Z = z, R' = r'),
\]

where

\[
P(Z = z, R' = r') = \alpha_k \prod_{i=1}^{k-1} \alpha_i \beta_i,
\]

and

\[
\alpha_1 = P(Z_1 = z_1) = \frac{mz_1 + n(1-z_1)}{mp + nq} p^{z_1} q^{1-z_1}, \quad z_1 = 0 \text{ or } 1,
\]

\[
\beta_1 = P(R'_1 = r'_1 | Z_1 = z_1) = \left( \frac{m-m_1}{r_1} \right) \left( \frac{n-n_1}{n-r'_1} \right)
\]

\[r'_1 = \max(0, r_1 + n_1 - n), \ldots, \min(r_1, m - m_1),\]

and for \(i = 2, \ldots, k, z_i = 0 \text{ or } 1,
\]

\[
\alpha_i = P(Z_i = z_i | Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}, R'_1 = r'_1, \ldots, R'_{i-1} = r'_{i-1})
\]

\[= \left( \frac{m-m_{i-1}-\sum_{j=1}^{i-1} r'_j}{r_i} \right) z_i + \left( n-n_{i-1}-\sum_{j=1}^{i-1} r''_j \right) (1-z_i)
\]

\[= \left( \frac{m-m_{i-1}-\sum_{j=1}^{i-1} r'_j}{r_i} \right) p + \left( n-n_{i-1}-\sum_{j=1}^{i-1} r''_j \right) q
\]

\[r'_i = \max(0, r_i + n_i + \sum_{j=1}^{i-1} r''_j - n), \ldots, \min(r_i, m - m_i - r'_{i-1} - \cdots - r'_i),\]

and for \(i = 2, \ldots, k-1, \) with \(r'_0 \equiv r''_0 \equiv 0,
\]

\[
\beta_i = P(R'_i = r'_i | Z_1 = z_1, \ldots, Z_i = z_i, R'_1 = r'_1, \ldots, R'_{i-1} = r'_{i-1})
\]

\[= \left( \frac{m-m_{i-1}-\sum_{j=1}^{i-1} r'_j}{r_i} \right) \left( \frac{n-n_{i-1}-\sum_{j=1}^{i-1} r''_j}{n-r'_i} \right)
\]

\[r'_i = \max(0, r_i + n_i + \sum_{j=1}^{i-1} r''_j - n), \ldots, \min(r_i, m - m_i - r'_{i-1} - \cdots - r'_i),\]

where \(p = P(Z = 1)\) and \(q = 1 - p.\)
3. Expected value of $M_k$

With the knowledge of the expected value of $M_k = \sum_{i=1}^{k} Z_i$, the number of $X$-failures in $W^k$ presents data of interest. In this section, by using Theorem 3.1, we will be able to approximate the expected value of $M_k$. We will need the Taylor expansion of the function in several variables (see, Adams (2007)).

For the $d$-dimensional function $T(x_1, \ldots, x_d)$ we have

$$T(x_1, \ldots, x_d) = T(a_1, \ldots, a_d) + \sum_{i=1}^{d} (x_i - a_i) T_{x_i}(a_1, \ldots, a_d) + \mathfrak{R}_2,$$

where $T_{x_i}$ denotes the partial derivative with respect to $x_i$ and $\mathfrak{R}_2$ is the remainder, which reduces to the tangent plane approximation

$$T(x_1, \ldots, x_d) \approx T(x_1, \ldots, x_d) = T(a_1, \ldots, a_d) + \sum_{i=1}^{d} (x_i - a_i) T_{x_i}(a_1, \ldots, a_d),$$

where $\mathfrak{T}(x_1, \ldots, x_d)$ is the Taylor polynomial of $T(x_1, \ldots, x_d)$ of degree 1 near the point $(a_1, \ldots, a_d)$.

In the special case, when $d = 2$ and the function depends on two variables, $x_1$ and $x_2$, the Taylor polynomial near the point $(a_1, a_2)$ is:

$$T(x_1, x_2) = T(a_1, a_2) + (x_1 - a_1) T_{x_1}(a_1, a_2) + (x_2 - a_2) T_{x_2}(a_1, a_2) + \mathfrak{R}_2$$

and

$$T(x_1, x_2) \approx T(x_1, x_2) = T(a_1, a_2) + (x_1 - a_1) T_{x_1}(a_1, a_2) + (x_2 - a_2) T_{x_2}(a_1, a_2).$$

From (2)--(6), we have

$$\mu_{Z_1} \equiv E(Z1) = \frac{m}{mp + nq} p \quad \text{(8)}$$

$$\mu_{Z'_1 \mid Z_1} \equiv E(R'_1 \mid Z_1 = z_1) = \frac{r_1 (m - m_1)}{(N - 1)} \quad \text{(9)}$$

Conditioning with respect to $Z_1$ and $Z_2$ gives

$$\mu_{Z_2 \mid Z_1, Z'_1} \equiv E(Z_2 \mid Z_1 = z_1, R'_1 = r'_1) = \frac{(m - m_1 - r'_1)}{(m - m_1 - r'_1) p + (n - n_1 - r'_1) q} \quad \text{(10)}$$

$$\mu_{Z'_2 \mid Z_2, Z'_1} \equiv E(R'_2 \mid Z_2 = z_2, R'_1 = r'_1) = \frac{r_2 (m - m_2 - r'_1)}{(N - r'_1 - 2)} \quad \text{(11)}$$

Continuing this process we obtain

$$\mu_{Z_i \mid Z_{i-1}, Z'_{i-1}} \equiv E \left( Z_i \mid Z_1 = z_1, \ldots, Z_{i-1} = z_{i-1}, R'_1 = r'_1, \ldots, R'_{i-1} = r'_{i-1} \right) = \frac{(m - m_{i-1} - \sum_{j=1}^{i-1} r'_j)}{(m - m_{i-1} - \sum_{j=1}^{i-1} r'_j) p + (n - n_{i-1} - \sum_{j=1}^{i-1} r''_j) q} \quad \text{p, } i = 3, \ldots, k \quad \text{(12)}$$

and

$$\mu_{R'_i \mid Z_i, Z'_{i-1}} \equiv E \left( R'_i \mid Z_1 = z_1, \ldots, Z_i = z_i, R'_{i-1} = r'_{i-1} \right) = \frac{r_i \left( m - m_i - \sum_{j=1}^{i-1} r'_j \right)}{(N - \sum_{j=1}^{i-1} r'_j - i)} \quad \text{p, } i = 3, \ldots, k \quad \text{ (13)}$$
From (8) and (9), it can be seen immediately that

\[ \mu_{Z_1} = E(Z_1) = \frac{m}{mp + nq} p \]

\[ \mu_{K_i} = \mu_{Z_1} \left[ E_{R_i} \left( K_i | Z_1 = z_1 \right) \right] = \frac{r_1 \left( m - \frac{m}{mp + nq} p \right)}{(N - 1)} \cdot \]

However, it can be observed from (10)–(13), that it is not easy to get exact expressions for \( \mu_{Z_i} \) and \( \mu_{K_i} \) for \( i = 2, 3, \ldots, k \). By using the Taylor expansion of the function of several variables, we can approximate them. For approximating \( \mu_{Z_2} \), we define

\[ \psi(z_1, r'_1) = \mu_{Z_2|Z_1 = z_1} = E \left( Z_2 | Z_1 = z_1, R'_1 = r'_1 \right) \]

\[ \approx \frac{(m - m_1 - r'_1) p + (n - n_1 - r'_1) q}{(m - m_1 - r'_1) p + (n - n_1 - r'_1) q} \cdot \]

The function \( \psi(z_1, r'_1) \) depends on two variables, \( z_1 \) and \( r'_1 \). We use the Taylor expansion for the function \( \psi(z_1, r'_1) \) substituting \( (a_1, a_2) = (\mu_{Z_1}, \mu_{K_1}) \) in (7). Thus, we have

\[ \approx \psi(z_1, r'_1) = \frac{m - \mu_{Z_1} - \mu_{K_1}}{m - \mu_{Z_1} - \mu_{K_1}} \cdot \]

By applying a repeated conditional expectation formula, we have

\[ \approx \mu_{Z_2} = \mu_{Z_1} \left[ E_{K_1} \left( \psi(z_1, r'_1) \right) \right] \]

\[ = \frac{(m - \mu_{Z_1} - \mu_{K_1}) p + (n - (1 - \mu_{Z_1}) - (r_1 - \mu_{K_1})) q}{(m - \mu_{Z_1} - \mu_{K_1}) p + (n - (1 - \mu_{Z_1}) - (r_1 - \mu_{K_1})) q} \cdot \]

Since the expected values after 2 terms vanish, when approximating \( \mu_{K_i} \) this method yields

\[ \approx \mu_{K_i} = \frac{r_i \left( m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j} \right)}{(N - \sum_{j=1}^{i-1} \mu_{K_j})} \cdot \]

Thus, by using similar considerations, we get the approximate values of \( \mu_{Z_i} \) and \( \mu_{K_i} \) for \( i = 3, 4, \ldots, k \) as follows:

\[ \approx \mu_{Z_i} = \frac{(m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j}) p + (n - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j} - (r_i - \mu_{K_i})) q}{(m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j}) p + (n - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j} - (r_i - \mu_{K_i})) q} \cdot \]

And

\[ \approx \mu_{K_i} = \frac{r_i \left( m - \sum_{j=1}^{i-1} \mu_{Z_j} - \sum_{j=1}^{i-1} \mu_{K_j} \right)}{(N - \sum_{j=1}^{i-1} \mu_{K_j} - i)} \cdot \]

Therefore, the approximate value for the expected value of the number of failures from \( X \)'s is

\[ E(M_k) \approx \sum_{i=1}^{k} \mu_{Z_i}, \quad (14) \]
3.1. Some special distributions

To evaluate the expected value of \( M_k \) for special distributions such as Gamma, Pareto and Weibull, we need to compute the value of \( p \) given in (15) for these distributions.

**Gamma distribution.** Let \( X \) and \( Y \) be independent random variables with Gamma p.d.f.'s \( \Gamma(\alpha_1, \beta_1) \) and \( \Gamma(\alpha_2, \beta_2) \), respectively. The p.d.f.'s of \( X \) and \( Y \) are

\[
f(x; \alpha_1, \beta_1) = \frac{x^{\alpha_1-1}}{\Gamma(\alpha_1)\beta_1^\alpha} \exp\left(-\frac{x}{\beta_1}\right) I(x > 0)
\]

and

\[
g(x; \alpha_2, \beta_2) = \frac{x^{\alpha_2-1}}{\Gamma(\alpha_2)\beta_2^\alpha} \exp\left(-\frac{x}{\beta_2}\right) I(x > 0),
\]

where \( I \) is the indicator function. It is not difficult to verify that, in this case,

\[
P_g = P(Z = 1) = P(X < Y) = \frac{I(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)},
\]

where \( B(a, b) \) is the beta function and \( I(a, b) \) is the incomplete beta function.

The expected value of \( M_k \) in the case where \( X \)'s have Gamma\( (\alpha_1, \beta_1) \) and \( Y \)'s have Gamma\( (\alpha_2, \beta_2) \) distributions, respectively, under the setup of the JPC scheme, can be calculated using (14) by substituting \( p = P_g \).

**Pareto distribution.** Let \( X \) and \( Y \) be independent random variables with Pareto p.d.f.'s \( \text{Pareto}(\lambda_1, \mu_1) \) and \( \text{Pareto}(\lambda_2, \mu_2) \), respectively. The p.d.f.'s of \( X \) and \( Y \) are

\[
f(x; \lambda_1, \mu_1) = \frac{\lambda_1 \mu_1}{x^{\lambda_1+1}} I(x > \mu_1)
\]

and

\[
g(x; \lambda_2, \mu_2) = \frac{\lambda_2 \mu_2}{x^{\lambda_2+1}} I(x > \mu_2).
\]

In this case, we have

\[
P_p = P(Z = 1) = P(X < Y) = \left\{ \begin{array}{ll}
1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \left( \frac{\mu_1}{\mu_2} \right)^{\lambda_1}, & \text{if } \mu_1 < \mu_2 \\
\frac{\lambda_1}{\lambda_1 + \lambda_2} \left( \frac{\mu_2}{\mu_1} \right)^{\lambda_2}, & \text{if } \mu_1 > \mu_2.
\end{array} \right.
\]

The expected value of \( M_k \) for two Pareto distributions under the setup of the JPC scheme can be evaluated from (14) by substituting \( p = P_p \).

**Weibull distribution.** Let \( X \) and \( Y \) be independent random variables with corresponding Weibull p.d.f.'s, \( \text{Weibull}(\beta_1, \alpha_1) \) and \( \text{Weibull}(\beta_2, \alpha_2) \), where \( \alpha_i, i = 1, 2 \), are the scale parameters and \( \beta_i, i = 1, 2 \), are the shape parameters. The p.d.f.'s of \( X \) and \( Y \) are given as

\[
f(x; \beta_1, \alpha_1) = \frac{\beta_1}{\alpha_1} \left( \frac{x}{\alpha_1} \right)^{\beta_1-1} \exp\left[-\left(\frac{x}{\alpha_1}\right)^{\beta_1}\right] I(x > 0)
\]

and

\[
g(x; \beta_2, \alpha_2) = \frac{\beta_2}{\alpha_2} \left( \frac{x}{\alpha_2} \right)^{\beta_2-1} \exp\left[-\left(\frac{x}{\alpha_2}\right)^{\beta_2}\right] I(x > 0).
\]

In this case, we have

\[
P_w = P(Z = 1) = P(X < Y) = \left\{ \begin{array}{ll}
1 - \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{\alpha_2}{\alpha_1} \right)^{\beta_1} \Gamma\left(\frac{\beta_1}{\beta_2}j + 1\right), & \text{if } \beta_1 < \beta_2 \\
\sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left( \frac{\alpha_2}{\alpha_1} \right)^{\beta_1} \Gamma\left(\frac{\beta_1}{\beta_2}j + 1\right), & \text{if } \beta_1 > \beta_2.
\end{array} \right.
\]

The expected value of \( M_k \) for these two Weibull distributions under the JPC scheme can be found by substituting \( p = P_w \) in (14).
4. Computation results and simulation study

In this section, some numerical results on calculation of the expected value of $M_k$ are presented. We have considered different sample sizes for the two populations from different distributions and censoring schemes. The expected values for $M_k$, calculated from the formula (14) for Gamma, Weibull and Pareto distributions mentioned above, are presented in Tables 1–3 in the Appendix.

Table 1 consists of the approximate values of $E(M_k)$—the expected number of failures from $X$’s in the JPC scheme, for observations $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ from populations having Gamma($\alpha_1, \beta_1$) and Gamma($\alpha_2, \beta_2$) distributions respectively, with the censoring scheme $R = (r_1, \ldots, r_k)$. Similarly, in Table 2, the numerical values of $M_k$ for two Pareto($\lambda_1, \mu_1$) and Pareto($\lambda_2, \mu_2$) distributions and in Table 3 for two Weibull($\beta_1, \alpha_1$) and Weibull($\beta_2, \alpha_2$) distributions are given.

It is important to examine the effectiveness of the proposed method in approximating the expected values of the number of failures. To study the performance of this approach, we also simulate 1,000 JPC samples from different distributions and different combinations of $N$, $k$, $m$, $n$, $p$ and censoring schemes $R$ and obtain the simulated expected values of the number of failures. The simulation results are also shown in Tables 1–3. The approximate expected values of the number of failures are close to the simulated expected values of the number of failures, for different distributions and censoring cases. Hence, this method is reliable where it is necessary to obtain the approximate values of the expected values of the number of failures.

In practical applications, it is very important for the researcher to have information about the expected values of the number of failures for the two populations before the experiment. Because in statistical inference procedures for parameters the number of failures for a population is directly related to estimators efficiency, information about the expected values of the number of failures for the two populations before the experiment is extremely important in the selection of a sampling plan.

From Table 1, for example, for $X$ and $Y$ having Gamma distributions with parameters ($\alpha_1, \beta_1$) = (4, 12), ($\alpha_2, \beta_2$) = (6, 4), respectively, and $p = 0.166$, $k = 10$, $m = 15$, $n = 15$, if we choose the censoring scheme $(0, \ldots, 0, 20)$, then we expect that approximately 2 failure times will be from the $X$ sample. In the case of the censoring scheme $(4, 0, 4, 0, 4, 0, 4, 0, 4, 0)$, we will expect approximately 4 observations of failure times from the $X$ sample.

From Table 2, for $X$ and $Y$ with Pareto distributions with parameters ($\lambda_1, \mu_1$) = (3, 6) and ($\lambda_2, \mu_2$) = (4, 4), respectively, for $m = 50$, $n = 50$, $p = 0.085$, $k = 50$, if the ordinary joint Type-II scheme is selected as $(0, \ldots, 0, 50)$, then approximately 8 observations of failure times are expected from the $X$ sample, whereas for the JPC scheme $(50, 0, \ldots, 0)$, this number increases to 25. Therefore, for inference procedures for the parameters of $X$ sample, we should prefer this scheme.

From Table 3, with $X$ having Weibull distribution with parameters ($\beta_1, \alpha_1$) = (5, 3) and $Y$ having Weibull distribution with parameters ($\beta_2, \alpha_2$) = (5, 5) we observe that for $m = 10$, $n = 20$, $k = 10$, $p = 0.928$ in an ordinary joint Type-II censoring scheme $(0, 0, \ldots, 0, 20)$, approximately 8 observations are expected from the $X$ sample and 2 from the $Y$ sample. However, if we choose the JPC scheme $(10, 10, 0, \ldots, 0)$ we would expect 5 observations of failure times from the $X$ sample and 5 from the $Y$ sample. This means that if $m$ and $n$ are small and $q$ less than $p$, and we employ an ordinary Type-II censoring scheme, then it is quite likely that we will observe very few failures from the second population in our censored sample, thus resulting in poor inference for the parameters of the second population. A JPC scheme balances this situation and results in failures from both populations, thus yielding more accurate inference for both parameters of the two populations. For example, Fig. 1, below, shows the approximate expected values of the number of failures with respect to $p \in (0, 1)$ for
Table 1
The Approximation of the Expected (A.E.) and Simulated Expected (S.E.) values of $M_k$ for some small, moderate and large values of $m, n, k$ and $p$ for two Gamma populations when joint Type-II progressive censoring is implemented.

<table>
<thead>
<tr>
<th>Scheme no.</th>
<th>$N$</th>
<th>$k$</th>
<th>Scheme(R)</th>
<th>$(m, n)$</th>
<th>$(\alpha_1, \alpha_2, \beta_1, \beta_2) = (4, 6, 12, 4), p = 0.166$</th>
<th>$(\alpha_1, \alpha_2, \beta_1, \beta_2) = (2, 5, 7, 3), p = 0.580$</th>
<th>$(\alpha_1, \alpha_2, \beta_1, \beta_2) = (3, 5, 2, 3), p = 0.904$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>56</td>
<td>10</td>
<td>(0, ..., 0, 20)</td>
<td>(15, 15, 15)</td>
<td>2.037 2.103 5.664 5.607 8.724 8.722</td>
<td>2.037 2.103 5.664 5.607 8.724 8.722</td>
<td>10.184 10.312 14.144 14.104 17.972 17.941</td>
</tr>
<tr>
<td>8</td>
<td>56</td>
<td>10</td>
<td>(0, ..., 0, 20)</td>
<td>(30, 30, 30)</td>
<td>12.002 11.939 18.470 18.392</td>
<td>12.002 11.939 18.470 18.392</td>
<td>20.000 20.000 30.000 30.000</td>
</tr>
</tbody>
</table>

$N = 30, k = 10, m = 20, n = 10$ for two cases $R = (0, ..., 0, 20)$ (joint Type-II censoring) and $R = (20, 0, ..., 0)$ (JPC) schemes. It is easy to see that the JPC scheme balances the situation.
Table 2
The Approximation of the Expected (A.E.) and Simulated Expected (S.E.) values of $M_k$ for some small, moderate and large values of $m, n, k$ and $P_p$ for two Pareto populations when joint Type-II progressive censoring is implemented.

<table>
<thead>
<tr>
<th>Scheme no.</th>
<th>$N$</th>
<th>$k$</th>
<th>$(m, n)$</th>
<th>$(\lambda_1, \lambda_2, \mu_1, \mu_2) = (3, 4, 6, 4.6)$</th>
<th>$p = 0.085$</th>
<th>$(\lambda_1, \lambda_2, \mu_1, \mu_2) = (4, 2, 7, 5)$</th>
<th>$p = 0.340$</th>
<th>$(\lambda_1, \lambda_2, \mu_1, \mu_2) = (3, 7, 3, 5)$</th>
<th>$p = 0.849$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>30</td>
<td>10</td>
<td>(0, 0, 0, 0)</td>
<td>2.352, 2.366, 3.877, 3.345</td>
<td>0.093</td>
<td>2.352, 2.366, 3.877, 3.345</td>
<td>0.093</td>
<td>2.352, 2.366, 3.877, 3.345</td>
<td>0.093</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>10</td>
<td>(5, 5, 0, 5)</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
</tr>
<tr>
<td>3</td>
<td>30</td>
<td>10</td>
<td>(10, 10, 0, 0)</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>10</td>
<td>(20, 20, 0, 0)</td>
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<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>10</td>
<td>(30, 30, 0, 0)</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
</tr>
<tr>
<td>6</td>
<td>30</td>
<td>10</td>
<td>(40, 40, 0, 0)</td>
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<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
</tr>
<tr>
<td>7</td>
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<td>10</td>
<td>(50, 50, 0, 0)</td>
<td>0.959, 0.947, 1.537, 1.565</td>
<td>0.093</td>
<td>0.959, 0.947, 1.537, 1.565</td>
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</tr>
</tbody>
</table>

Acknowledgements
The authors are grateful to two anonymous referees, the Co-Editor and Associate Editor for the careful reading and constructive suggestions, which were helpful in improving the presentation.
Table 3
The Approximation of the Expected (A.E.) and Simulated Expected (S.E.) values of $M_k$ for some small, moderate and large values of $m, n, k$ and $P_y$ for two Weibull populations when joint Type-II progressive censoring is implemented.

<table>
<thead>
<tr>
<th>Scheme no.</th>
<th>$N$</th>
<th>$k$</th>
<th>Scheme(R)</th>
<th>$(m, n)$</th>
<th>$(f_1, f_2, \alpha_1, \alpha_2) = (4, 5, 3, 2), p = 0.160$</th>
<th>$(f_1, f_2, \alpha_1, \alpha_2) = (6, 2, 3, 3), p = 0.431$</th>
<th>$(f_1, f_2, \alpha_1, \alpha_2) = (5, 5, 3, 5), p = 0.928$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>A.E. $M_k$</td>
<td>S.E. $M_k$</td>
<td>A.E. $M_k$</td>
<td>S.E. $M_k$</td>
</tr>
<tr>
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<td>(20, 10)</td>
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<td>3.543</td>
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<td>5.042</td>
<td>7.389</td>
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<td>5.402</td>
<td>5.504</td>
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<td>5.927</td>
<td>6.029</td>
<td>8.389</td>
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<td>8.764</td>
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<td>9.619</td>
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<td>10</td>
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<td>(15, 10)</td>
<td>10.367</td>
<td>10.470</td>
<td>14.659</td>
</tr>
<tr>
<td>11</td>
<td>20</td>
<td>10</td>
<td>(10, 0, 0, 0, 0, 0, 0, 0, 0, 0)</td>
<td>(20, 10)</td>
<td>11.216</td>
<td>11.319</td>
<td>16.039</td>
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<tr>
<td>12</td>
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<td>10</td>
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<td>(15, 10)</td>
<td>11.967</td>
<td>12.070</td>
<td>17.349</td>
</tr>
<tr>
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<td>20</td>
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<td>14.317</td>
<td>14.420</td>
<td>21.379</td>
</tr>
</tbody>
</table>

Appendix

Proof of Theorem 3.1. We can obtain $\alpha_i$ from the fact that in a population containing two members (one X and one Y), the probability of observing $X$ first is

$$p = P(Z = 1) = P(X < Y) = \int_{0}^{\infty} \tilde{G}(x; \theta_2) dF(x; \theta_1).$$  (15)
Now for the $m$ observations of $X$’s, $n$ observations of $Y$’s and $k$ times censoring using Bayes’ theorem with prior probabilities $p/(p + q)$ for $z = 1$ and $q/(p + q)$ for $z = 0$ and with mass probabilities $a/(a + b)$ for $z = 1$ and $b/(a + b)$ for $z = 0$ for each step, we have the posterior mass probabilities $ap/(ap + bq)$ for $z = 1$ and $bp/(ap + bq)$ for $z = 0$. Here in the $j$th step for derivation of probability

$$P(Z_j = z_i | Z_{1}, \ldots, Z_{j-1} = z_{j-1}, R_1' = r_1', \ldots, R_{j-1}' = r_{j-1}')$$

the numbers $a$ and $b$ are taken as $a_j = \left( m - m_{j-1} - \sum_{i=1}^{j-1} r_i' \right)$ and $b_j = \left( n - n_{j-1} - \sum_{i=1}^{j-1} r_i' \right)$.

Also, we can obtain $\beta_i$ from the fact that in the $i$th step, $R_i'$ has a hypergeometric distribution with parameters $m - m_i - \sum_{j=1}^{i-1} r_j'$, $n - n_i - \sum_{j=1}^{i-1} r_j'$ and $r_i$. □

Note that in the literature the probability $p = P(X < Y)$ is called stress–strength reliability. In recent years the problem of estimating $p$ has been given much attention in reliability and related fields. For a comprehensive review of results on estimating stress–strength reliability we refer Kotz et al. (2003).

References