Some distribution free properties of statistics based on record values and characterizations of the distributions through a record

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Abstract

Let $X_1, X_2, ..., X_n, ...$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.) with continuous distribution function (d.f.) $F$. Define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min \{ j : j > U(n-1), \; X_j > X_{U(n-1)} \}, \; n > 1$. Let $X_{U(n)}$ be upper record values, $n = 1, 2, ...$. Suppose that $X'_1, X'_2, ..., X'_n, ...$ is another sequence of i.i.d. r.v.-s with d.f. $F$. In this paper we investigate some statistics based on $X_{U(n)}$, $U(n)$ and $X'_1, X'_2, ..., X'_n, ...$. A characterization of the uniform distribution is given based on $X_{U(n)}$.

Key words and phrases. Order statistics, characteristic function, uniform distribution, record value, characterization.

1. INTRODUCTION

Let $X_\infty = (X_1, X_2, ..., X_n, ...)$ be an infinite sample from continuous distributions with d.f. $F(x)$, $0 < F(x) < 1$. $X = [X_\infty]_n$ is the first $n$ coordinates. As infinite sample, $X_\infty$, we consider an element of sample space $(\mathbb{R}^\infty, B^\infty, P^\infty)$, where $\mathbb{R}^\infty$ is the space of $(X_1, X_2, ..., X_n, ...)$ sequences; the $\sigma-$algebra, $B^\infty$, is a $\sigma-$algebra arising from sets $\bigcap_{j=1}^N \{ X_j \in B_j \}$, $B_j \in B$, $N = 1, 2, ...$, such that $B$ is the Borel $\sigma-$algebra over the subset of $\mathbb{R}$; and $P^\infty$ is a probability measure according to $F(x)$ distribution function on $(\mathbb{R}^\infty, B^\infty)$. Here the symbol $[.]_n$ denotes the operator of projection from $\mathbb{R}^\infty$ in $\mathbb{R}^n$.

Let $U(n)$ and $X_{U(n)}$ be upper record times and record values, respectively. It is known that the d.f. of record value is (see Ahsanullah, 1995)
\[ F_n(x) = P \{ X_{U(n)} \leq x \} = \frac{1}{(n-1)!} \int_{-\infty}^{x} \left[ \ln \frac{1}{1 - F(u)} \right]^{n-1} dF(u), -\infty < x < \infty. \]


Suppose that \( X'_{\infty} = (X'_{1}, X'_{2}, ..., X'_{n}, ...) \) be another infinite sample from the distribution with d.f. \( F \) and \( X'_{\infty} \) is obtained independently from \( X_{\infty} \).

Let \( X'_{\infty} \) denote \( X'_{1}, X'_{2}, ..., X'_{m} \). We are interested in the behavior of \( X'_{\infty} \) related to \( X_{U(r)} \), \( r = 1, 2, ... \).

2. DISTRIBUTION FREE PROPERTIES

Consider \( X_{U(1)}, X_{U(2)}, ..., X_{U(n)}, ..., \) and \( X'_{1}, X'_{2}, ..., X'_{m} \).

**Lemma 2.1.** For any \( k = 1, 2, ..., m \) and \( r = 1, 2, ... \) it is true that

\[ P \{ X'_{k} < X_{U(r)} \} = 1 - \frac{1}{2^r}. \]

**Proof.**

\[
P \{ X'_{k} < X_{U(r)} \} = \frac{1}{(r-1)!} \int_{-\infty}^{\infty} F(u) \left[ \ln \frac{1}{1 - F(u)} \right]^{r-1} dF(u)
= \frac{1}{(r-1)!} \int_{0}^{1} u \left[ \ln \frac{1}{1 - u} \right]^{r-1} du = \frac{1}{(r-1)!} \int_{0}^{\infty} y^{r-1}(1 - e^{-y})e^{-y} dy = 1 - \frac{1}{2^r}.
\]

(2.1)

**Corollary 2.1.** For any \( k, r = 1, 2, ... \) and \( s > r \), it is true that

\[ P \{ X_{U(r)} < X'_{k} < X_{U(s)} \} = \frac{1}{2^r} - \frac{1}{2^s}. \]

(2.2)

Let us define the following r.v. for given \( r \):

\[ \xi_{i}(r) = 1 \text{ if } X'_{i} < X_{U(r)} \text{ and } \xi_{i}(r) = 0 \text{ if } X'_{i} \geq X_{U(r)}, \quad i = 1, 2, ..., m \]

and denote \( S_{m}(r) = \sum_{i=1}^{m} \xi_{i}(r) \). It is clear that \( S_{m}(r) \) is the number of observations \( X'_{1}, X'_{2}, ..., X'_{m} \) which are less than \( X_{U(r)} \). Note that the random variables \( \xi_{1}(r), \xi_{2}(r), ..., \xi_{m}(r) \) are generally dependent, therefore, we do not have Bernoulli trials here.

**Theorem 2.1.** For any \( m, r = 1, 2, ... \)

\[ P \{ S_{m}(r) = k \} = \frac{\binom{m}{k}}{(r-1)!} \int_{0}^{\infty} e^{-z(m-k+1)}(1 - e^{-z})^{k}z^{r-1} dz \]

(2.3)
Lemma 2.1 we have

\[ \mathbb{E}[\xi] \]

First of all we compute the expected value and the variance of \( S \)

probability (2.5), hence

\[ \text{Var}(A) = 1, \]

where \( A = \{ k = 0, 1, 2, \ldots, m \} \) and \( \bar{A} = \{ k = 0, 1, 2, \ldots, m \} \) denote the complements of event \( A \).

It is clear that

\[ P \{ S_m(r) = k \} = \sum_{i_1, i_2, \ldots, i_m} P \{ A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \cap \bar{A}_{i_{k+1}} \cap \bar{A}_{i_{k+2}} \cap \ldots \cap \bar{A}_{i_m} \} \]

(2.4)

where \( A_{i_k} = \{ X_{i_k} < X_U(r) \} \), \( k = 0, 1, 2, \ldots, m \) and \( \bar{A}_{i_k} \) denotes the complements of event \( A_{i_k} \). One has

\[ P \{ A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \cap \bar{A}_{i_{k+1}} \cap \bar{A}_{i_{k+2}} \cap \ldots \cap \bar{A}_{i_m} \} = \]

\[ P \{ X_{i_1} < X_U(r), \ldots, X_{i_k} < X_U(r), X_{i_{k+1}} \geq X_U(r), \ldots, X_{i_m} \geq X_U(r) \} = \]

\[ = \frac{1}{(r-1)!} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \ln \frac{1}{1 - F(v)} \right]^{r-1} dF(u_1) dF(u_2) \ldots dF(u_m) dF(v) \]

\[ = \frac{1}{(r-1)!} \int_{-\infty}^{\infty} (1 - F(v))^{m-k} F^k(v) \left[ \ln \frac{1}{1 - F(v)} \right]^{r-1} dF(v) \]

\[ = \frac{1}{(r-1)!} \int_{0}^{1} (1 - y)^{m-k} y^k \left[ \ln \frac{1}{1 - y} \right]^{r-1} dy. \]

(2.5)

The number of summands in (2.4) is equal to \( \binom{m}{k} \) and all of them have the same probability (2.5), hence

\[ P \{ S_m(r) = k \} = \frac{\binom{m}{k}}{(r-1)!} \int_{0}^{1} (1 - y)^{m-k} y^k \left[ \ln \frac{1}{1 - y} \right]^{r-1} dy \]

\[ = \frac{\binom{m}{k}}{(r-1)!} \int_{0}^{\infty} e^{-z(m-k+1)}(1 - e^{-z})^k z^{r-1} dz. \quad (Q.E.D.) \]

3. THE ASYMPTOTIC DISTRIBUTIONS

Now, we investigate the behavior of the distribution, \( S_m(r) \), for large \( m \).

First of all we compute the expected value and the variance of \( S_m(r) \). By using Lemma 2.1 we have

\[ ES_i(r) = 1 - \frac{1}{2^r}, \quad i = 1, 2, \ldots, m. \]

Hence \( E S_m(r) = m(1 - \frac{1}{2^r}) \).

For the variance of \( S_m(r) \) we get

\[ \text{var}((S_m(r)) = E(\sum_{i=1}^{m} \xi_i(r))^2 - m^2(1 - \frac{1}{2^r})^2 = \sum_{i=1}^{m} E(\xi_i(r))^2 + 2 \sum_{i<j} E(\xi_i(r))\xi_j(r) \]

\[ - m^2(1 - \frac{1}{2^r})^2. \]

(3.1)

It is clear that
\begin{align*}
E(\xi_i(r))^2 &= 1 - \frac{1}{2^r} \quad E(\xi_j(r))^2 = 1 - \frac{1}{2^r}. 
\end{align*}

Let us consider
\begin{align*}
E\xi_i(r)\xi_j(r) &= P\{X'_i < X_U(r), X'_j < X_U(r)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{u_1} \int_{-\infty}^{u_2} dF(u_1)dF(u_2)dF(r) = \\
&= \frac{1}{(r-1)!}\int_{-\infty}^{\infty} F^2(v) \left[ \ln \frac{1}{1-F(v)} \right]^{r-1} dF(v) = \frac{1}{(r-1)!}\int_{0}^{1} x^2 \left[ \ln \frac{1}{1-x} \right]^{r-1} dx \\
&= 1 - \frac{2}{2^r} + \frac{1}{3^r}.
\end{align*}

Consequently we obtain
\begin{align*}
\sum_{i<j} E\xi_i(r)\xi_j(r) &= \frac{m(m-1)}{2} \left[ 1 - \frac{2}{2^r} + \frac{1}{3^r} \right]. 
\end{align*}

By using (3.2) and (3.3) in (3.1) we obtain
\begin{align*}
\text{var}(\langle S_m(r) \rangle) &= m^2(\frac{1}{3^r} - \frac{2}{2^r}) + m(\frac{1}{2^r} - \frac{1}{3^r}).
\end{align*}

Let us denote
\begin{align*}
S^*_m(r) &= S_m(r) - ES_m(r) / \sqrt{\text{var}(S_m(r))}. 
\end{align*}

Denote \( a = \frac{1}{2^r}, b = \sqrt{\frac{1}{3^r} - \frac{1}{2^r}} \).

\textbf{Theorem 3.1.} A statistic \( S^*_m(r) \) has a continuous limiting distribution as \( m \to \infty \), with probability density function \( f^* \) defined as follows
\begin{align*}
f^*(x) &= \begin{cases} 
\frac{b}{(r-1)!} \left[ \ln \frac{1}{a-bx} \right]^{r-1} & \text{if } x \in \left[ \frac{a-1}{b}, \frac{a}{b} \right] \\
0 & \text{if } x \notin \left[ \frac{a-1}{b}, \frac{a}{b} \right].
\end{cases}
\end{align*}

\textbf{Proof.} Consider the characteristic function of \( S_m(r) \):
\begin{align*}
\varphi_m(t) &= E\exp(itS_m(r)) = \sum_{k=0}^{m} e^{itk} P\{S_m(r) = k\} \\
&= \frac{1}{(r-1)!}\sum_{k=0}^{m} e^{itk} \binom{m}{k} \int_{0}^{\infty} e^{-z(m-k)} e^{-z(1 - e^{-z})k} z^{r-1} dz \\
&= \frac{1}{(r-1)!}\int_{0}^{\infty} z^{r-1} e^{-z} \left\{ \sum_{k=0}^{m} \binom{m}{k} [e^{it(1 - e^{-z})}]^k e^{-z(m-k)} \right\} dz \\
&= \frac{1}{(r-1)!}\int_{0}^{\infty} z^{r-1} e^{-z} [e^{it(1 - e^{-z})} + e^{-z}]^m dz
\end{align*}
\[
\frac{1}{(r-1)!} \int_0^\infty z^{r-1} e^{-z} \left[ 1 + (e^{it} - 1)(1 - e^{-z}) \right]^m dz = H\left[ 1 + (e^{it} - 1)(1 - e^{-z}) \right]^m, \tag{3.4}
\]

where
\[
H(f) = \frac{1}{(r-1)!} \int_0^\infty z^{r-1} e^{-z} f(z) dz.
\]

It is clear that the functional \(H(f)\) has the following properties:

1. \(H(1) = 1\)
2. \(H(c_1 f_1 + c_2 f_2) = c_1 H(f_1) + c_2 H(f_2)\), \(c_1, c_2 = \text{const.}\)

Denote \(\varphi_m^*(t) = E \exp(itS_m)\). We clearly obtain

\[
\varphi_m^*(t) = \exp(-\frac{itES_m}{\sqrt{\text{var}(S_m)}}) \varphi_m(t) = \exp(-\frac{itES_m}{\sqrt{\text{var}(S_m)}}) \times
\]

\[
H\left[ 1 + (\exp(\frac{it}{\sqrt{\text{var}(S_m)}}) - 1)(1 - \exp(-z)) \right]^m = H(\exp(-\frac{itES_m}{\sqrt{\text{var}(S_m)}}) \times
\]

\[
\times \left[ 1 + (\exp(\frac{it}{\sqrt{\text{var}(S_m)}}) - 1)(1 - \exp(-z)) \right]^m. \tag{3.5}
\]

Denote

\[
g_m(t) = \exp(-\frac{itES_m}{\sqrt{\text{var}(S_m)}}) \left[ 1 + (\exp(\frac{it}{\sqrt{\text{var}(S_m)}}) - 1)(1 - \exp(-z)) \right]^m
\]

\[
\ln g_m(t) = -\frac{itES_m}{\sqrt{\text{var}(S_m)}} + m \ln \left[ 1 + (\exp(\frac{it}{\sqrt{\text{var}(S_m)}}) - 1)(1 - \exp(-z)) \right]
\]

\[
= -\frac{itES_m}{\sqrt{\text{var}(S_m)}} + m \ln \left[ 1 + (\exp(\frac{it}{\sqrt{\text{var}(S_m)}}) + o(\frac{t}{\sqrt{\text{var}(S_m)}}))(1 - \exp(-z)) \right]
\]

\[
= -\frac{itES_m}{\sqrt{\text{var}(S_m)}} + \frac{itm(1 - \exp(-z))}{\sqrt{\text{var}(S_m)}} + O\left(\frac{1}{m}\right).
\]

As a result it becomes from above as

\[
g_m(t) = \exp(it\frac{m(1 - \exp(-z)) - ES_m}{\sqrt{\text{var}(S_m)}}) + O\left(\frac{1}{m}\right). \tag{3.6}
\]

By taking (3.6) into consideration in (3.5) we have

\[
H(g_m(t)) = H(\exp(it\frac{m(1 - \exp(-z)) - ES_m}{\sqrt{\text{var}(S_m)}})) + O\left(\frac{1}{m}\right),
\]

and by having the limit

\[
\lim_{m \to \infty} \varphi_m^*(t) = \lim_{m \to \infty} H(g_m(t)) = \lim_{m \to \infty} H(\exp(it\frac{m(1 - \exp(-z)) - m(1 - 2^{-\frac{1}{2}})}{\sqrt{m^2(\frac{1}{2\pi} - \frac{1}{2\pi}) + m(\frac{1}{2\pi} - \frac{1}{2\pi})}})) =
\]

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\[ \varphi(t) = \lim_{m \to \infty} \varphi^*_m(t). \]

One can prove that \( \varphi(t) \) is continuous at the point \( t = 0 \). In fact, by using the expansion

\[ e^t = 1 + t + \frac{t^2}{2} + o(t^2) \]

one obtains from (3.7)

\[ \varphi(t) = \exp(it \frac{a - e^{-z}}{b}) = 1 + \frac{a - e^{-z}}{b} \cdot it - \frac{(a - e^{-z})^2}{2b^2} t^2 + o(t^2). \]  

(3.8)

From (3.7) and (3.8) it follows

\[ \varphi(t) = 1 - \frac{t^2}{2} + o(t^2). \]

Thus, if \( t \to 0 \), then

\[ |\varphi(t) - \varphi(0)| = \left| 1 - \frac{t^2}{2} + o(t^2) - 1 \right| = \frac{t^2}{2} + o(t^2) \to 0. \]

Let \( F_m(x) \) be the d.f. of the statistic \( S^*_m(r) \), where

\[ x = \frac{k - E(S^*_m(r))}{\sqrt{\text{var}(S^*_m(r))}}, \quad k = 0, 1, 2, \ldots, m. \]

Using the Levy-Cramer theorem for characteristic functions (see Petrov, 1975, Theorem 10, P.15) we obtain that \( F_m(x) \to F(x) \) as \( m \to \infty \), \( x \in \left[ \frac{a-1}{b}, \frac{a}{b} \right] \) and \( F \) has a characteristic function \( \varphi(t) \), that is

\[ \varphi(t) = \int_{\frac{a-1}{b}}^{\frac{a}{b}} e^{itx} dF(x). \]

In other hand if we change the variable \( \frac{a}{b} - \exp(-z) = x \) in (3.7) we have

\[ \varphi(t) = H(e^{it \frac{a - \exp(-z)}{b}}) = \frac{1}{(r-1)!} \int_0^\infty z^{r-1} e^{-z} e^{it \frac{a - \exp(-z)}{b}} \, dz \]

\[ = \frac{b}{(r-1)!} \int_{\frac{a-1}{b}}^{\frac{a}{b}} \left[ \ln \frac{1}{a - bx} \right]^{r-1} e^{itx} \, dx. \]
Hence \( F(x) \) has the density function

\[ f^*(x) = \frac{b}{(r-1)!} \left( \ln \frac{1}{a-bx} \right)^{r-1} \quad \text{if} \quad x \in \left[ \frac{a-1}{b}, \frac{a}{b} \right] \quad \text{and} \quad f^*(x) = 0 \quad \text{if} \quad x \notin \left[ \frac{a-1}{b}, \frac{a}{b} \right]. \]

The following theorem clearly may be obtained from Theorem 3.1. However we adduce a different proof which is interesting in our opinion.

**Theorem 3.2.** It is true that,

\[
\lim_{m \to \infty} \sup_{0 \leq x \leq 1} \left| P \left\{ \frac{S_m(r)}{m} \leq x \right\} - \frac{1}{(r-1)!} \int_0^x \left[ \ln \left( \frac{1}{1-u} \right) \right]^{r-1} du \right| = 0.
\]

**Proof.** We have

\[
S_m(r) = \sum_{i=1}^{m} \xi_i(r) = \sum_{i=1}^{m} I_{\{(-\infty,X_{U(r)})\}} (X_i'), \tag{3.9}
\]

where \( I_{\{A\}}(x) = 1 \) if \( x \in A \) and \( I_{\{A\}}(x) = 0 \) if \( x \notin A \). Using the representation (3.9) we may write

\[
P \left\{ \frac{S_m(r)}{m} \leq x \right\} = P \left\{ \frac{1}{m} \sum_{i=1}^{m} I_{\{(-\infty,X_{U(r)})\}} (X_i') \leq x \right\}
= P \left\{ \int_{-\infty}^{x} I_{\{(-\infty,X_{U(r)})\}} (u) dF_m^*(u) \leq x \right\}, \tag{3.10}
\]

where \( F_m^*(u) \) denotes the empirical distribution function of sample \( X_1', X_2', ..., X_m' \). Note that any infinite sample \( X_\infty = (X_1, X_2, ..., X_n, ...) \) may be considered also as a sequence of i.i.d. random variables \( X_1(\omega), X_2(\omega), ..., X_n(\omega), ... \) defined in probability space \( \{\Omega, \mathcal{F}, P\} \), where \( \Omega \) is a set of points, \( \mathcal{F} \) is a \( \sigma \)-field of subsets of \( \Omega \), and \( P \) is a probability distribution of the elements of \( \mathcal{F} \), \( F(x) = P \{ \omega : X_i(\omega) \leq x \} \). Denote

\[
G^*(F) = \int_{-\infty}^{x} I_{\{(-\infty,x)\}} (u) dF(u), \quad (x \text{ is fixed}) \tag{3.11}
\]

\[
G(F) = \int_{-\infty}^{\infty} I_{\{(-\infty,x)\}} (u) dF(u), \tag{3.12}
\]

where \( G(F) = G(F)(\omega) \) is a random variable defined in the probability space \( \{\Omega, \mathcal{F}, P\} \). Using (3.12), one can write (3.10) as follows

\[
P \left\{ \frac{S_m(r)}{m} \leq x \right\} = P \left\{ \int_{-\infty}^{\infty} I_{\{(-\infty,X_{U(r)})\}} (u) dF_m^*(u) \leq x \right\} = P \{ G(F_m^*) \leq x \}.
\]

The functional (3.11) is continuous according to uniform metric. One can follow from the Glivenko-Cantelli Theorem \( \left( P \left\{ \omega : \sup_{u} |F_m^*(u) - F(u)| \to 0 \right\} = 1 \right) \) \( G^*(F_m^*) \to G^*(F) \) almost sure (see Borovkov,1984). It is clear that

\[
P \left\{ \omega : \lim_{m \to \infty} G(F_m^*) = G(F) \right\}
\]
\[
\begin{align*}
&= \int_{-\infty}^{\infty} P \left\{ \lim_{r \to \infty} \int_{-\infty}^{x} I_{\{(-\infty, X_{U(r)})\}}(u) dF_m^*(u) = \int_{-\infty}^{x} I_{\{(-\infty, X_{U(r)})\}}(u) dF(u) / X_{U(r)} = x \right\} dF_r(x) \\
&= \int_{-\infty}^{\infty} P \left\{ \lim_{r \to \infty} \int_{-\infty}^{x} I_{\{(-\infty, x)\}}(u) dF_m^*(u) = \int_{-\infty}^{x} I_{\{(-\infty, x)\}}(u) dF(u) \right\} dF_r(x) \\
&= \int_{-\infty}^{\infty} P \{ \lim G^*(F_m^*) = G^*(F) \} dF_r(x) = 1,
\end{align*}
\]
where \( F_r(x) = P \{ X_{U(r)} \leq x \} \). So \( G(F_m^*) \to G(F) \) almost sure in \((\Omega, \mathcal{F}, P)\). Thus, \( G(F_m^*) \to G(F) \) in distribution. We have

\[
P \{ G(F) \leq x \} = P \left\{ \int_{-\infty}^{x} I_{\{(-\infty, X_{U(r)})\}}(u) dF(u) \leq x \right\} = P \left\{ \int_{-\infty}^{X_{U(r)}} dF(u) \leq x \right\}
\]

\[
P \{ F(X_{U(r)}) \leq x \} = \frac{1}{(r-1)!} \int_{0}^{x} \left[ \ln \left( \frac{1}{1-u} \right) \right]^{r-1} du.
\]

Above uniform convergence is the consequence of convergence in distribution and continuity of d.f. \( U_r(x) = \frac{1}{(r-1)!} \int_{0}^{x} \left[ \ln \left( \frac{1}{1-u} \right) \right]^{r-1} du \).

**Remark 3.1.** Let \( r = 1 \), \( X_{U(1)} = X_1 \). Then \( \xi_1(1) = 1 \) if \( X'_i < X_1 \) and \( \xi_i(1) = 0 \) if \( X'_i \geq X_1 \), \( i = 1, 2, ..., m \). And \( S_m(1) = \sum_{i=1}^{m} \xi_i(1) \) is the number of observations \( X'_1, X'_2, ..., X'_m \) which are less than \( X_1 \). From Theorem 3.2.

follows that

\[
\lim_{m \to \infty} \sup_{0 \leq x \leq 1} \left| P \left\{ \frac{S_m(1)}{m} \leq x \right\} - x \right| = 0.
\]

**Remark 3.2.** We may similarly obtain the following result for order statistics. Let \( X_1, X_2, ..., X_n \) be a sample from the distribution with continuous d.f. \( F \) and \( Y_1, Y_2, ..., Y_m \) be a sample from the distribution with continuous d.f. \( G \). Also let \( X_{(1)}, X_{(2)}, ..., X_{(n)} \) be order statistics constructed by \( X_1, X_2, ..., X_n \). Consider the hypothesis \( H_0 : F(u) = G(u) \). Let \( \xi_i = 1 \) if \( X_{(r)} \leq Y_i \leq X_{(s)} \) and \( \xi_i = 0 \) if \( X_{(r)} > Y_i \lor X_{(s)} < Y_i \), \( i = 1, 2, ..., m, \ 1 \leq r < s \leq n \). Denote \( \nu_m = \sum_{i=1}^{m} \xi_i \).

**Theorem 3.3.** If \( H_0 \) is true, then

\[
\sup_x \left| P \left\{ \frac{\nu_m}{m} \leq x \right\} - P \{ W_{rs} \leq x \} \right| \to 0, \text{ as } m \to \infty,
\]

where \( W_{rs} = F(X_{(s)}) - F(X_{(r)}) \).

It is known that \( W_{rs} \) has the probability density function (see David, 1970)

\[
f(w_{rs}) = \begin{cases} 
\frac{1}{B(s-r, n-s+r+1)} w_{rs}^{s-r-1} (1 - w_{rs})^{n-s+r} & \text{if } 0 \leq w_{rs} \leq 1 \\
0 & \text{otherwise} 
\end{cases}
\]

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4. CHARACTERIZATION OF UNIFORM DISTRIBUTION BY RECORDS

Let $X_\infty$ be an infinite sample from the distribution with d.f. $F(x)$, $X'$ denotes one observation from $F(x)$ and $X'$ does not depend on $X_\infty$. Denote by $\Lambda$ a class of continuous distribution functions $F(x), -\infty < x < \infty$ with property

$$F(x) \geq x, 0 < x < 1 \quad \text{or} \quad F(x) \leq x, 0 < x < 1.$$ 

For example let

$$F_n(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
x^n & \text{if } 0 < x < 1 \\
1 & \text{if } x \geq 1
\end{cases}, \quad G_n(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 - (1 - x)^n & \text{if } 0 < x < 1 \\
1 & \text{if } x \geq 1
\end{cases},$$

$n = 1, 2, \ldots$. Denote by $U_{a,b}(x)$ the d.f. of uniform distribution on $[a, b]$. It is clear that $F_n(x) \in \Lambda, G_n(x) \in \Lambda, n = 1, 2, \ldots$ and $U_{a,a+1}(x) \in \Lambda, a \in \mathbb{R}$.

**Theorem 4.1.** Let $F(x) \in \Lambda$. Then $F(x)$ is same with uniform distribution on $[0, 1]$ if and only if

$$EX_{U(n)} = 1 - 2^{-n} \text{ for some } n \geq 1. \quad (4.1)$$

**Proof.** Let $F(x) = U_{0,1}(x)$. Clearly one can write

$$EX_{U(n)} = \frac{1}{(n-1)!} \int_0^1 u \left( \ln \frac{1}{1-u} \right)^{n-1} du = 1 - \frac{1}{2n}. \quad (4.1)$$

To proof the rest of the Theorem suppose that $EX_{U(n)} = 1 - 2^{-n}$ for some $n \geq 1$. From the Lemma 2.1 it follows

$$1 - \frac{1}{2^n} = P \{ X' < X_{U(n)} \} = \frac{1}{(n-1)!} \int_{-\infty}^\infty F(x) \left( \ln \frac{1 - F(x)}{1} \right)^{n-1} dF(x)$$

It follows from (4.1) for some $n \geq 1$

$$\frac{1}{(n-1)!} \int_{-\infty}^\infty x \left( \ln \frac{1 - F(x)}{1} \right)^{n-1} dF(x) - \frac{1}{(n-1)!} \int_{-\infty}^\infty F(x) \left( \ln \frac{1 - F(x)}{1} \right)^{n-1} dF(x) =$$

$$= \frac{1}{(n-1)!} \int_{-\infty}^\infty (x - F(x)) \left( \ln \frac{1 - F(x)}{1} \right)^{n-1} dF(x) = 0. \quad (4.2)$$

It follows with the assumption on $F$ that $F(x) - x$ has the same sign in the above integral within the integration limits. Let $F(x) \geq x, 0 < x < 1$. Then $F(x) = 1$ if $x \geq 1$. Therefore one can write the following expression with the help of (4.2)

$$\frac{1}{(n-1)!} \int_{-\infty}^1 (x - F(x)) \left( \ln \frac{1 - F(x)}{1} \right)^{n-1} dF(x)$$

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\[ + \frac{1}{(n-1)!} \int_0^1 (x - F(x)) \left[ \ln \frac{1}{1 - F(x)} \right]^{n-1} dF(x) = 0. \]

Hence \( F(x) = x \) if \( 0 < x < 1 \) and \( F(x) = 0 \) if \( x \leq 0 \). The case \( F(x) \leq x, \) \( 0 < x < 1 \) is investigated analoguously. (Q.E.D.)

5. APPENDIX

In this section we are giving some results in order to explain why we are considering two independent samples \( X_\infty \) and \( X'_\infty \) from the same distribution with the d.f. \( F \). Let us define \( X_\infty = (X_1, X_2, ..., X_n, ...) \) and \( X_U(r) \) be the \( r \) th upper record value. \( X_{U(r)+1}, X_{U(r)+2}, ..., X_{U(r)+m} \) are observations which comes after \( X_{U(r)} \). It is not difficult to see that, random variables \( X_{U(r)+1}, X_{U(r)+2}, ..., X_{U(r)+m} \) are mutually independent and identically distributed with d.f. \( F \), for any \( r \). Furthermore \( X_{U(r)+k} \) and \( X_{U(r)} \) are independent for \( k = 1, 2, ... \). In fact, (for simplicity, consider the case of \( r = 2 \))

\[
P\{X_{U(r)+1} \leq t\} = \sum_{i=2}^{\infty} P\{X_{U(r)+1} \leq t, U(2) = i\}
\]

\[
= \sum_{i=2}^{\infty} P\{X_{i+1} \leq t, X_2 \leq X_1, ..., X_{i-1} \leq X_1, X_i > X_1\}
\]

\[
= \sum_{i=2}^{\infty} P\{X_{i+1} \leq t\} P\{X_2 \leq X_1, ..., X_{i-1} \leq X_1, X_i > X_1\}
\]

\[
= P\{X_{i+1} \leq t\} \sum_{i=2}^{\infty} P\{U(2) = i\} = P\{X_{i+1} \leq t\} = F(t).
\]

Then from the Lemma 2.1, we have

\[
P\{X_{U(r)+1} < X_{U(r)}\} = 1 - \frac{1}{2r}.
\]

With the analogy let \( X_{U(r)+1}, X_{U(r)+2}, ..., X_{U(r)+m} \) are observations which comes after \( X_{U(r)} \). Then

\[
P\{X_{U(r)+1} < X_{U(r)}, ..., X_{U(r)+m} < X_{U(r)}\} = P\{S_m(r) = m\} = \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{1}{(i+1)r}.
\]

\((S_m(r) \text{ is defined in section 1}).

Remark 5.1. It is clear that in all theorems above we may replace \( X'_i \) by \( X_{U(r)+i} \).
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