IUE – MATH 306 – Abstract Algebra

1^{st} Midterm — April 4, 2013 — 15:00 - 16:50

Name: ____________________ **ID #:** ____________________

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Important: Show all your work. Answers without sufficient explanation might not get full credit. Be neat.

Signature: ____________________

GOOD LUCK!

1. Which of the following binary structures are group? Why?

a) $G = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ and $(x, y) * (z, w) = (x + z, y - w)$.

b) $G = \mathbb{R}\backslash\{1\}$ and $x * y = x + y - xy$.

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a) $\langle G, *\rangle$ is not a group because the binary operation $*$ is not associative. Indeed, for $(x, y), (0, z), (0, 1) \in G$,

$$[(x, y) * (0, z)] * (0, 1) = (x, y - z) * (0, 1) = (x, y - z - 1),$$
$$(x, y) * [(0, z) * (0, 1)] = (x, y) * (0, z - 1) = (x, y - z + 1),$$

and they are not equal.

b) Let $x, y, z \in G$. Then

$$\begin{aligned} x * (y * z) &= x * (y + z - yz) \\ &= x + y + z - yz - x(y + z - yz) \\ &= x + y + z - yz - xy - xz + xyz \\ &= x + y - xy + z - (x + y - xy)z \\ &= (x + y - xy) * z \\ &= (x * y) * z. \end{aligned}$$

Thus, the operation $*$ is associative. Since $x*0 = x+0-x0 = x = 0+x-0x = 0 * x$ for all $x \in G$, it follows that 0 is identity element for $*$. For any $x \in G$, we try to find inverse element x^{-1} of x in G; i.e., $x * x^{-1} = 0$. Then $x + x^{-1} - xx^{-1} = 0$ implies that $x^{-1} = \frac{-x}{1-x}$. Since $x \neq 1$, $\frac{-x}{1-x} \in G$. Note that also $x^{-1} * x = 0$. Therefore, every element of G has an inverse element in G. Hence, $\langle G, *\rangle$ is a group.

2. Let, $\emptyset \neq B \subseteq A$ and $b \in B$. Which of the following subsets are subgroups? Why?

a) $\{\sigma \in S_A \,|\, \sigma(b) = b\}$

b) $\{\sigma \in S_A \,|\, \sigma(b) \in B\}$

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a) Let $H := \{\sigma \in S_A \,|\, \sigma(b) = b\}$, and let $\sigma, \tau \in H$. Then $\sigma(b) = b$ and $\tau(b) = b$. So, $\sigma\tau(b) = \sigma(b) = b$. Thus, $\sigma\tau \in H$. It means that H is closed under the permutation multiplication. Since $\sigma(b) = b$, it follows that $\sigma^{-1}(b) = \sigma^{-1}\sigma(b) = b$. Therefore, $\sigma^{-1} \in H$. Clearly, identity element 1 of S_A is in H because $1(b) = b$. Hence, H is a subgroup of S_A.

b) Let $K := \{\sigma \in S_A \,|\, \sigma(b) \in B\}$, and let $\sigma, \tau \in K$ with $\sigma(b) = b' \in B$ such that $\tau(b') \notin B$ (Note that since B is a proper subset of A, we can find such an element b'). Then $\tau\sigma(b) = \tau(b') \notin B$. It implies that K is not closed under the permutation multiplication. Therefore, K is not a subgroup of S_A.

3. Let $G = A_4$ be the alternating group on 4 letters, and $H \leq G$.

a) Write all possible values of $|H|$. Briefly explain your reasoning.

b) Explicitly find a proper subgroup H of maximal order.

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a) Note that the order of G is $\frac{4!}{2} = 12$. By Lagrange theorem, since $H \leq G$, $|H| \mid 12$. Hence all possible values of $|H|$ are $1, 2, 3, 4, 6, 12$.

b) The 12 elements of G are identity element 1, the 3-cycles $(123), (132), (124), (142), (134), (143), (234), (243)$ of order 3 and product of two odd permutations $(12)(34), (13)(24), (14)(23)$ of order 2.

Since the subgroup of order 12 is all of G, we start with order 6. Such a subgroup H consists of elements of order 1, 2, or 3 by Lagrange theorem. Note that $1 \in H$. Since G has only 3 elements of order 2, H must contain at least one 3-cycle (abc). Then the inverse of cycle (abc), (acb) is also in H because $H \leq G$. If H has an element of the form $(ab)(cd)$, by closure property of subgroup, $(abc)(ab)(cd) = (ac)(bd) \in H$ and $(acb)(ab)(cd) = (bcd) \in H$. Moreover, $(bcd)^{-1} = (bdc) \in H$. Again by closure property, $(ab)(cd)(ac)(bd) = (ad)(bc) \in H$ and $(ad)(bc)(bcd) = (adc) \in H$. So, $(adc)^{-1} = (acd) \in H$, $(acd)(ac)(bd) = (adb) \in H$ and $(adb)^{-1} = (abd) \in H$. In conclusion, we have $H = G$. Thus, H can not consists of an element in G of order 2, i.e. of the form $(ab)(cd)$. If H consists of only 3-cycles, it need 6 of them. However, multiplication of two 3-cycle except with their inverses gives an element of order 2: $(abc)(abd) = (ac)(bd)$, $(abc)(acd) = (bcd)$ and $(abc)(bcd) = (ab)(cd)$. It means that H does not consist of only 3-cycles. Hence, we can not find a subgroup of G of order 6.

Now, we will try to find a subgroup of order 4. By Lagrange theorem, such a group H consists of elements of order dividing 4. So, its elements have order 1, 2 or 4. We give the elements of G above, explicitly. G has no elements of order 4, also H has no elements of order 4. Thus, the elements $(12)(34), (13)(24), (14)(23)$ of order 2 and identity element 1 are possible elements of H. Consider $H = \{1, (12)(34), (13)(24), (14)(23)\}$. Since order of every non-identity elements is 2, the inverse of each element is itself. Clearly, it is closed under permutation multiplication:

$$
\begin{aligned}
(12)(34)(13)(24) &= (13)(24)(12)(34) = (14)(23),\\
(12)(34)(14)(23) &= (14)(23)(12)(34) = (13)(24),\\
(13)(24)(14)(23) &= (14)(23)(13)(24) = (12)(34).
\end{aligned}
$$

Hence, H is subgroup of G. Also, it is the proper subgroup of G of maximal order.

4. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 6 & 4 & 1 & 9 & 2 & 5 & 7 & 8 \end{pmatrix}$.

a) Write σ in cycle notation.

b) Find orders of all disjoint cycles of σ.

c) Find the order of σ.

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a) $\sigma = (134)(26)(5987)$.

b) Let $\tau = (134)$, $\alpha = (26)$ and $\beta = (5987)$. Then $\tau^2 = (143)$ and $\tau^3 = 1$. So, the order of τ is 3. $\alpha^2 = 1$ implies that the order of α is 2. Consider β; $\beta^2 = (58)(97)$, $\beta^3 = (5789)$ and $\beta^4 = 1$. Thus, the order of β is 4. Remember that an n-cycle is of order n.

c)

$$\begin{aligned}
\sigma^2 &= (143)(58)(97), \\
\sigma^3 &= (26)(5789), \\
\sigma^4 &= (134), \\
\sigma^5 &= (143)(26)(5987), \\
\sigma^6 &= (58)(97), \\
\sigma^7 &= (134)(26)(5789), \\
\sigma^8 &= (143), \\
\sigma^9 &= (26)(5987), \\
\sigma^{10} &= (134)(58)(97), \\
\sigma^{11} &= (143)(26)(5789), \\
\sigma^{12} &= 1.
\end{aligned}$$

Therefore, the order of σ is 12. Note that $|\sigma|$ =lcm$(3, 2, 4)$, where it is the least common multiple of order of disjoint cycles.

5. Show that if G is a finite abelian group having exactly 3 subgroups, then G is cyclic.

By fundamental theorem of finitely generated abelian groups, G is isomorphic to a direct product of cyclic groups. Suppose that G has two or more generators in each generating set. Let g and g' be two generators in a fixed generating set. Since both are generators in the same generating set, $\langle g \rangle \neq \langle g' \rangle$ and these cyclic groups are proper non-trivial subgroups of G. Then G has 4 different subgroups: 1, G, $\langle g \rangle$ and $\langle g' \rangle$. It is contradiction because G has exactly 3 subgroups. Hence, G is generated by one element and it is isomorphic to a cyclic group by fundamental theorem of finite abelian groups.