MATH 306 – HOMEWORK ASSIGNMENT II

1. Prove that a non-cyclic group of order p^2 has exactly $p+3$ subgroups. (Hint: use Lagrange theorem.)

2. Let H be a subgroup of a group G. Show that for all $x \in G$, $x^2 \in H$ implies that $H \trianglelefteq G$ and G/H is abelian group.
(Hint: consider $(gh)^2 h^{-1}(g^{-1})^2$.)

3. Let $G = \langle a \rangle \times \langle b \rangle$, where $|a| = 8$ and $|b| = 4$.

- **a.** Find all pairs $x \in \langle a \rangle, y \in \langle b \rangle$ satisfying $G = \langle x \rangle \times \langle y \rangle$.
- **b.** Let $H = \langle a^2 b, b^2 \rangle \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Prove that there are no elements $x, y \in G$ such that $G = \langle x \rangle \times \langle y \rangle$ and $H = \langle x^2 \rangle \times \langle y^2 \rangle$. (Hint: use x and y found in part **a.**)

4. Let G be a group, and let $N, H \trianglelefteq G$ such that $G = NH$. Prove that $G/(N \cap H) \cong (G/N) \times (G/H)$.

5. Let G be a nontrivial abelian group whose normal subgroups are only itself and trivial subgroup 1. Prove that G is isomorphic to a cyclic group of prime order.

Due Date: April 28, 2014.