MATH 306 – HOMEWORK SOLUTIONS II

1. Let G be a non-cyclic group and $|G| = p^2$. Let g be any element of G. By Lagrange theorem, $|g| \mid p^2$. Thus 1, p, p^2 are only possibilities for the order of g. If $|g| = p^2$, then $G = \langle g \rangle$, and so G will be cyclic. But it will be contradiction because G is non-cyclic group. Therefore, every nonidentity element g of G has order p. Since G has $p^2 - 1$ elements different from identity e and for each $g \neq e$, $\langle g \rangle$ has $p - 1$ elements different from identity, G has $\frac{p^2-1}{p-1} = p + 1$ many subgroups of order p. With trivial subgroup 1 and itself, G has $p + 3$ subgroups.

2. Suppose that for all $x \in G$, $x^2 \in H$. Let $g \in G$ and $h \in H$. Then $(gh)^2 h^{-1}(g^{-1})^2 = ghghh^{-1}g^{-1}g^{-1} = ghg^{-1}$. By assumption, $(gh)^2 \in H$ and $(g^{-1})^2 \in H$. Thus, $ghg^{-1} \in H$. Then H is normal subgroup of G.

Note that if g is an element of G, then $(gH)^2 = g^2H = H$ because $g^2 \in H$. We want to show that $gHg'H = g'HgH$ for all $g, g' \in G$; i.e., $gHg'H(gH)^{-1}(g'H)^{-1} = H$. Since $(gH)^2 = H$ for all $g \in G$, we have $gHg'H(gH)^{-1}(g'H)^{-1} = gHg'HgHg'H = gg'gg'H = (gg')^2H = H$ by assumption. Hence, G/H is abelian group.

3. **a.** Note that $|G| = |\langle a \rangle||\langle b \rangle| = 8.4 = 32$. Since $\langle x \rangle \leq \langle a \rangle$ and $\langle y \rangle \leq \langle b \rangle$, by Lagrange theorem, $|x||8$ and $|y||4$. However, $|\langle x \rangle||\langle y \rangle| = |\langle x \rangle \times \langle y \rangle| = |G| = 32$. Therefore, $|x| = 8$ and $|y| = 4$. Hence, x can be a, a^3, a^5, a^7 (because 1, 3, 5, 7 are relatively prime with 8), and y can be b, b^3 (indeed, 1 and 3 are relatively prime with 4).

b. Suppose that we have such elements $x, y \in G$. By exercise **a.**, x^2 is either a^2 or a^6, and y^2 must be b^2. Therefore, such a subgroup $H = \langle x^2 \rangle \times \langle y^2 \rangle$ is generated as either $\langle a^2 \rangle \times \langle b^2 \rangle$ or $\langle a^6 \rangle \times \langle b^2 \rangle$. However, $H = \langle a^2b, b^2 \rangle$ and we can not obtain a^2b using the generators a^2, b^2 and their inverses a^6, b^2. Hence, we can not have such elements x and y.

4. We define a map $\Phi : G \to (G/N) \times (G/H)$ by

$$\Phi(g) = (gN, gH)$$

for all $g \in G$. Then Φ is a homomorphism. Indeed, for all $g_1, g_2 \in G$,

$$\begin{aligned}\Phi(g_1 g_2) &= (g_1 g_2 N, g_1 g_2 H)\\ &= (g_1 N g_2 N, g_1 H g_2 H)\\ &= (g_1 N, g_1 H)(g_2 N, g_2 H)\\ &= \Phi(g_1)\Phi(g_2).\end{aligned}$$

Now, consider $\ker \Phi$. Since for given $g \in G$, $(gN, gH) = (N, H)$ if and only if $g \in N \cap H$, it follows that $\ker \Phi = N \cap H$. Now, we will show that Φ is surjective. Let $(g_1 N, g_2 H)$ be an element in $(G/N) \times (G/H)$. Then $G = NH$ implies that $g_1 = n_1 h_1$ and $g_2 = n_2 h_2$ for some $n_1, n_2 \in N$ and $h_1, h_2 \in H$. Since H is normal subgroup of G, $n_1 h_1 = h_1' n_1$ for some $h_1' \in H$. So,

$$\begin{aligned}(g_1 N, g_2 H) &= (h_1' n_1 N, n_2 h_2 H)\\ &= (h_1' N, n_2 H).\end{aligned}$$

Choose $g := h_1' n_2$. Since H is normal in G, $h_1' n_2 = n_2 h_1''$ for some $h_1'' \in H$. Hence,

$$\begin{aligned}\Phi(g) &= (h_1' n_2 N, n_2 h_1'' H)\\ &= (h_1' N, n_2 H)\\ &= (g_1 N, g_2 H).\end{aligned}$$

It means Φ is surjective. By the fundamental homomorphism theorem, $G/(N \cap H) = G/\ker \Phi \cong (G/N) \times (G/H)$.

5. Let g be a nonidentity element of G. Since each subgroup of an abelian group is normal, $\langle g \rangle$ is normal subgroup of G. By assumption, $G = \langle g \rangle$. Therefore, G is cyclic group. By Theorem 6.10, either $G \cong \mathbb{Z}$ or $G \cong \mathbb{Z}_n$ for $n \in \mathbb{N}$ (in fact, $n = |g|$). Since $\mathbb{Z}$ has nontrivial proper subgroups, G can not be isomorphic to $\mathbb{Z}$ and so, $G \cong \mathbb{Z}_n$. However, if n is not prime, $\mathbb{Z}_n$ has a nontrivial proper subgroup. So, G is isomorphic to $\mathbb{Z}_p$ for some prime p.