IUE – MATH 306 – Abstract Algebra

2$^{\text{nd}}$ Midterm — May 9, 2014 — 15:00 - 16:50

Name: ____________________ **ID #:** ____________________

Q1	Q2	Q3	Q4	Q5	TOTAL
20	20	20	20	20	100

Important: Show all your work. Answers without sufficient explanation might not get full credit. Be neat.

Signature: ____________________

GOOD LUCK!

1. Write the definition of
a) kernel,
b) normal subgroup.

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a) Let $\phi : G \to G'$ be a group homomorphism from a multiplicative group G with identity e into a multiplicative group G' with identity e'. Then

$$\text{Ker}\phi = \{g \in G \,|\, \phi(g) = e'\}$$

b) Let H be a subgroup of a group G. Then H is called normal subgrouop of G if $gH = Hg$ for all $g \in G$.

2. Show that A_n is a normal subgroup of S_n.

Note that the index $[S_n : A_n]$ of A_n in S_n is 2 because $|S_n| = n!$ and $|A_n| = n!/2$. It means that number of left (or right) cosets of A_n in S_n are 2. Let $\sigma \in S_n - A_n$. Then the two left cosets of A_n is $1A_n = A_n$ and σA_n. So, $\sigma A_n = S_n - A_n$. On the other hand, the two right cosets of A_n is A_n and $A_n\sigma$. Thus, we must have $A_n\sigma = S_n - A_n$. Hence, $\sigma A_n = A_n\sigma$.

3. Find all possible group homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$. For each homomorphism, describe its kernel and its image.

Note that $\mathbb{Z} \times \mathbb{Z}$ is generated by $(1,0)$ and $(0,1)$. Let ϕ be a group homomorphism from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$ and (m,n) an element in $\mathbb{Z} \times \mathbb{Z}$. Then

$$\begin{aligned}\phi((m,n)) &= \phi((1,0) + \cdots + (1,0) + (0,1) + \cdots + (0,1)) \\ &= \phi((1,0)) + \cdots + \phi((1,0)) + \phi((0,1)) + \cdots + \phi((0,1)) \\ &= m\phi((1,0)) + n\phi((0,1)).\end{aligned}$$

Therefore, it is enough to define that $\phi((1,0))$ and $\phi((0,1))$. For elements $(a,b),(c,d) \in \mathbb{Z} \times \mathbb{Z}$, let $\phi_{(a,b),(c,d)} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be the homomorphism with

$$\phi_{(a,b),(c,d)}((1,0)) = (a,b)$$
$$\phi_{(a,b),(c,d)}((0,1)) = (c,d).$$

Then $\phi((m,n)) = (ma + nc, mb + nd)$ for any $(m,n) \in \mathbb{Z} \times \mathbb{Z}$. We consider four cases:

(a) $\mathrm{Ker}\phi_{(a,b),(c,d)} \neq \{(0,0)\}$ and $\mathrm{Im}\phi_{(a,b),(c,d)} \neq \mathbb{Z} \times \mathbb{Z}$.
  (i) If $a = b = c = d = 0$, then $\phi_{(0,0),(0,0)}$ is zero map, and its kernel is $\mathbb{Z} \times \mathbb{Z}$, its image is 0.
  (ii) If $a = c = 0$, then $\phi_{(0,b),(0,d)}((m,n)) = (0, mb + nd)$. Then it is clearly neither injective nor surjective.
  (iii) Vice versa for $b = d = 0$.

(b) $\mathrm{Ker}\phi_{(a,b),(c,d)} = \{(0,0)\}$ and $\mathrm{Im}\phi_{(a,b),(c,d)} \neq \mathbb{Z} \times \mathbb{Z}$.
  (i) If $b = c = 0$ and $a, d \neq 0$, then $\phi_{(a,0),(0,d)}((m,n)) = (ma, nd)$. Since $a, d \neq 0$, $(ma, nd) = (m'a, n'd)$ implies $m = m'$ and $n = n'$. So, $\phi_{(a,0),(0,d)}$ is injective. However, it is surjective if and only if $a, d = \pm 1$. Indeed, we can find a pair (m,n) satisfying $ma = 1$ and $nd = 1$ if and only if $a, d = \pm 1$.
  (ii) Vice versa for $a = d = 0$ and $b, c \neq 0$.

(c) $\mathrm{Ker}\phi_{(a,b),(c,d)} \neq \{(0,0)\}$ and $\mathrm{Im}\phi_{(a,b),(c,d)} = \mathbb{Z} \times \mathbb{Z}$.
There is no possibility satisfying that $\phi_{(a,b),(c,d)}$ is surjective but not injective.

(d) $\mathrm{Ker}\phi_{(a,b),(c,d)} = \{(0,0)\}$ and $\mathrm{Im}\phi_{(a,b),(c,d)} = \mathbb{Z} \times \mathbb{Z}$.
Since $\phi(m,n) = (ma + nc, mb + nd)$,

$$\begin{aligned}\mathrm{Im}\phi &= \langle (a,c),(b,d) \rangle \\ \mathrm{Ker}\phi &= \{(m,n) \,|\, ma + nc = 0 \wedge mb + nd = 0\} \\ &= \{(m,n) \,|\, ma + nc = 0\} \wedge \{(m,n) \,|\, mb + nd = 0\}.\end{aligned}$$

So, it is enough to find one of them explicitly. Note that if $a = c = 0$, then clearly each (m,n) is in this set. If $a = 0$ and $c \neq 0$ (or vice versa), then the solution set is $(m,0)$ (or $(0,n)$). Thus, wlog take $a, c \neq 0$. Let $d = \gcd(a,c)$ and $a = dk$, $c = d\ell$ for some $k, \ell \in \mathbb{Z}$. Hence, $ma + nc = 0$ implies $mdk + nd\ell = 0$, and so, $mk + n\ell = 0$. Since $\gcd(k,\ell) = 1$, $m = \ell\alpha$ and $n = -k\alpha$ for some $\alpha \in \mathbb{Z}$. Hence the solution set is $\{(\ell\alpha, -k\alpha) | \alpha \in \mathbb{Z}\} = \langle (\ell, -k) \rangle$

4. Give examples of rings described below or explain why it does not exist;

a) a ring containing zero divisors but not unity,

b) a ring with a unique zero divisor.

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a) There is a ring containing zero divisors but not unity. Consider $2\mathbb{Z}_4$. We know that $2\mathbb{Z}_4$ is an abelian group under addition, the multiplication operation has associative and the distribution property and $2\mathbb{Z}_4$ has no unit. However, it has zero divisors. For example, $(\overline{2})(\overline{4}) = \overline{0}$ in $2\mathbb{Z}_4$.

b) There is a ring with a unique zero divisor. If R is a ring and $0 \neq a$ is a zero divisor of R, then there exists a nonzero element b in R such that $ab = 0$. Thus, b is also a zero divisor of R. If R has unique zero divisor, $a = b$ and $a^2 = 0$ in R. For example, $\mathbb{Z}_4$ is a ring with unique zero divisor 2.

5. Show that characteristic of an integral domain is either zero or a prime number.

Let R be an integral domain. Suppose that the characteristic of R is $n \neq 0$. Then n is the smallest positive integer satisfying $n \cdot a = 0$ for all $a \in R$. Therefore, $n \cdot 1 = 0$, where $0 \neq 1$ is unit element of integral domain R. If $n = xy$ for some $x, y \in R$, then $(x \cdot 1)(y \cdot 1) = (xy) \cdot 1 = n \cdot 1 = 0$. Since R is integral domain, one of $x \cdot 1$ or $y \cdot 1$ must be 0. However, n is the smallest positive integer satisfying this property. Hence, n must be a prime number.