25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 250-Linear Algebra and Differential Equations for Engineers 15.12.2015

İzmir University of Economics Faculty of Arts and Sciences, Department of Mathematics

Midterm Exam 2

Student Name and Surname:..............................

Instructor's Name:..............................

(1) (*a*) Let

$$A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

.

Determine if w is in Col A? Is w in Nul A?

$$Aw = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -12+12 \\ -6+6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad w \in \text{Nul } A$$

$$\begin{bmatrix} -6 & 12 & 2 \\ -3 & 6 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1+R_2 \to R_2} \begin{bmatrix} -6 & 12 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

the system is consistent so w is in Col A.

(1) (*b*) Find the characteristic polynomial and the eigenvalues of the matrix

$$A = \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix}$$

$$|A - \lambda I| = \left| \begin{bmatrix} 7 & -2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix}$$

$$(7-\lambda)(3-\lambda)+4 = 21 - 10\lambda + \lambda^2 + 4 = 0$$

$\lambda^2 - 10\lambda + 25 = 0$ chr polyn.

$s: -10$, $p: 25$ $\Big\} \begin{matrix} -5 \\ -5 \end{matrix}$ $\quad (\lambda-5)(\lambda-5) = 0$

$\lambda_{1,2} = 5$ repeated eigenvalue

(2) (*a*) Find an orthogonal basis for the column space of the matrix (Use Gram-Schmidt process).

$$A = \begin{array}{c} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix} \\ \;\; x_1 \quad x_2 \quad x_3 \end{array}$$

$v_1 = x_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$

$v_2 = x_2 - p$

$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} \cdot v_1$

$x_3 \cdot v_1 = 4 - 2 + 2 + 0 = 4.$

$v_1 \cdot v_1 = 4.$

$x_3 \cdot v_2 = \cancel{4} \cdot \frac{-5}{\cancel{2}} - 2 \cdot \frac{5}{2} + \cancel{2} \cdot \frac{5}{\cancel{2}} + 0$

$= -10 - \cancel{5} + \cancel{5} + 0 = -10.$

$v_2 \cdot v_2 = \frac{25}{4} + \frac{25}{4} + \frac{25}{4} + \frac{25}{4} = \frac{100}{4} = 25$

$x_2 \cdot v_1 = -1 + 4 + 4 + -1 = 6.$

$v_1 \cdot v_1 = 1 + 1 + 1 + 1 = 4.$

$v_2 = \begin{bmatrix}-1\\4\\4\\-1\end{bmatrix} - \frac{6}{4} \cdot \begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}-1 - 3/2\\4 - 3/2\\4 - 3/2\\-1 - 3/2\end{bmatrix} = \begin{bmatrix}-5/2\\5/2\\5/2\\-5/2\end{bmatrix}$

$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} \cdot v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} \cdot v_2 = \begin{bmatrix}4\\-2\\2\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{\cancel{10}\,2}{\cancel{25}\,5}\begin{bmatrix}-5/2\\5/2\\5/2\\-5/2\end{bmatrix} = \begin{bmatrix}3\\-3\\1\\-1\end{bmatrix} + \begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\-2\\2\\-2\end{bmatrix}$

(*b*) Orthonormalize the three vectors found in (a) and use them as the column vectors of the matrix Q.

$Q = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix}$

$Q^T = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}$

(*c*) Find an upper triangular matrix R such that $A = QR$.

$R = Q^T \cdot A = \underset{3\times4}{\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}} \underset{4\times3}{\begin{bmatrix} 1 & -1 & 4 \\ 1 & 4 & -2 \\ 1 & 4 & 2 \\ 1 & -1 & 0 \end{bmatrix}}$

$R = \begin{bmatrix} 2 & 3 & 2 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{bmatrix}$

(3) (*a*) Find an LU-factorization of the coefficient matrix of the system:

$x + 2y + 3z = 6$

$2x - 3y + 2z = 14$

$3x + y - z = -2$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 5/7 & 1 \end{bmatrix}$$

$-2\ell_1 + \ell_2 \to \ell_2 \quad -3\ell_1 + \ell_3 \to \ell_3$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & \boxed{-7} & -4 \\ 0 & \boxed{-5} & -10 \end{pmatrix} \xrightarrow{-\frac{5}{7}\ell_2 + \ell_3 \to \ell_3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 0 & -\frac{50}{7} \end{pmatrix} = U$$

(*b*) Solve the above linear system by using LU-factorization.

$Lx = b$

$Ux = y$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{5}{7} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 14 \\ -2 \end{pmatrix}$$

$y_1 = 6$

$2 \cdot 6 + y_2 = 14 \Rightarrow y_2 = 2$

$3 \cdot 6 + \frac{5}{7} \cdot 2 + y_3 = -2$

$y_3 = -\frac{150}{7}$

$Ux = y$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & -7 & -4 \\ 0 & 0 & -\frac{50}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ -\frac{150}{7} \end{pmatrix}$$

$\boxed{x_3 = 3}$

$-7x_2 - 4x_3 = 2$

$-7x_2 = 14$

$\boxed{x_2 = -2}$

$x_1 + 2(-2) + 3 \cdot (3) = 6$

$x_1 = 6 - 9 + 4 = 1$

$\boxed{x_1 = 1}$

(4) (*a*) Find a least-squares solution of the inconsistent system $Ax = b$ for the given A and b

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$A^TA\,x = A^Tb$

$A^T.A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix}$

$A^T.A = \begin{pmatrix} 6 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad (A^TA)^{-1} = \frac{1}{18-1}\begin{bmatrix} 3 & -1 \\ -1 & 6 \end{bmatrix} = \begin{pmatrix} 3/17 & -1/17 \\ -1/17 & 6/17 \end{pmatrix}$

$A^T.b = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 1 & 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$

$\hat{x} = (A^TA)^{-1}.A^Tb = \begin{pmatrix} \frac{3}{17} & \frac{-1}{17} \\ \frac{-1}{17} & \frac{6}{17} \end{pmatrix} \cdot \begin{pmatrix} 8 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{24}{17} \\ \frac{-8}{17} \end{pmatrix}$ $\Rightarrow$ least-squares solution.

(*b*) Find the closest point to y in the subspace W spanned by

$$y = \begin{bmatrix} 3 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$v_1.v_1 = 2-1+0-1 = 0$

$\hat{y} = \frac{y_1.v_1}{v_1.v_1}.v_1 + \frac{y.v_2}{v_2.v_2}.v_2$

$y.v_1 = 6-1+6-1 = 10$

$v_1.v_1 = 4+1+9+1 = 15$

$y.v_2 = 3+1+0+1 = 5$

$v_2.v_2 = 1+1+1 = 3$

$\hat{y} = \frac{10}{15}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} + \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \begin{pmatrix} 3 \\ 1 \\ -2 \\ -1 \end{pmatrix}$