25 points	25 points	25 points	25 points	100 points
1	2	3	4	Total

MATH 250-Linear Algebra and Differential Equations for Engineers 03.10.2015

İzmir University of Economics Faculty of Arts and Sciences, Department of Mathematics

Midterm Exam 1

Student Name and Surname:...........................

Instructor's Name:...................................

(1) (*a*) Use Gaussian elimination to solve the linear system $Ax = b$ that is compute the row echelon form of the augmented matrix

$2x - 3y + 4z = -12$
$x - 2y + z = -5$
$3x + y + 2z = 1$

$$\begin{bmatrix} 2 & -3 & 4 & -12 \\ 1 & -2 & 1 & -5 \\ 3 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -2 & 1 & -5 \\ 2 & -3 & 4 & -12 \\ 3 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[-3R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 1 & 2 & -2 \\ 0 & 7 & -1 & 16 \end{bmatrix}$$

$$\xrightarrow{-7R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 1 & -5 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -15 & 30 \end{bmatrix}$$

$-15z = 30 \Rightarrow z = -2$

$y + 2z = -2 \Rightarrow y = 2$

$x - 2y + z = -5 \Rightarrow x = 1$

(1) (*b*) The augmented matrix is given below for a system of equations $Ax = b$, where A consists of the first three columns and b is the fourth column. If the system is consistent, find the general solution:

$$\begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 4 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{x_3 \text{ free.}}$

$x_1 + 2x_2 - 3x_3 = 5$

$x_2 + 4x_3 = -5$

$\boxed{x_2 = -4x_3 - 5}$

$x_1 = 5 - 2x_2 + 3x_3$

$5 - 2(-4x_3 - 5) + 3x_3$

$x_1 = 5 + 8x_3 + 10 + 3x_3$

$\boxed{x_1 = 15 + 11x_3}$

(2) (*a*) Let

$$a_1 = \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ -10 \\ 6 \end{bmatrix}$$

Determine whether b can be written as a linear combination of a_1 and a_2.

$x_1 . a_1 + x_2 . a_2 = b$

$x_1 . \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix} + x_2 . \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \\ 6 \end{bmatrix}$

$$\begin{bmatrix} 3 & -4 & 2 \\ 4 & 1 & -10 \\ -4 & 1 & 6 \end{bmatrix} \xrightarrow[\frac{4}{3}R_1 + R_3 \to R_3]{-\frac{4}{3}R_1 + R_2 \to R_2} \begin{bmatrix} 3 & -4 & 2 \\ 0 & 19/3 & -38/3 \\ 0 & -13/3 & 26/3 \end{bmatrix} \xrightarrow{\frac{13}{19}R_2 + R_3 \to R_3}$$

$$\begin{bmatrix} 3 & -4 & 2 \\ 0 & 19/3 & -38/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$3x_1 - 4x_2 = 2$

$\frac{19}{3}x_2 = \frac{-38}{3} \Rightarrow \boxed{x_2 = -2}$

$\boxed{x_1 = -2}$

(2) (*b*) Let

$$v_1 = \begin{bmatrix} 1 \\ -3 \\ 5 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -3 \\ 8 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix}$$

Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 8 & -2 \\ 5 & 3 & -6 \end{bmatrix} \xrightarrow[-5R_1+R_3 \to R_3]{3R_1+R_2 \to R_2} \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 4 \\ 0 & 18 & -16 \end{bmatrix} \xrightarrow{18R_2+R_3 \to R_3} \begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 4 \\ 0 & 0 & 56 \end{bmatrix} \xrightarrow{-R_2 \to R_2}$$

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & -4 \\ 0 & 0 & 56 \end{bmatrix}$$

V_1, V_2, V_3 basic variables.
no free variable.
$Ax=0$ has only trivial soln.
So the columns of A are linearly indep.

(3) (*a*) Let

$$A = \begin{bmatrix} 1 & x & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 4 \\ y \end{bmatrix}. \text{ If } AB = \begin{bmatrix} 12 \\ 6 \end{bmatrix}$$

find x and y.

$1 \cdot 2 + 4 \cdot x + 3 \cdot y = 12$ $\qquad 2 + 4x + 18 = 12$ $\qquad 4x + 20 = 12$

$4 - 4 + y = 6 \rightarrow \boxed{y = 6}$ $\qquad 4x = -8.$

$\boxed{x = -2}$

(3) (b) Let

$$A = \begin{bmatrix} 2 & -3 & 2 \\ 3 & -1 & -2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 3 \\ 2 & -2 \end{bmatrix}$$

Find $A^T + 2B$.

$$A^T = \begin{bmatrix} 2 & 3 \\ -3 & -1 \\ 2 & -2 \end{bmatrix} \qquad 2B = \begin{bmatrix} 0 & 2 \\ 2 & 6 \\ 4 & -4 \end{bmatrix} \qquad A^T + 2B = \begin{bmatrix} 2 & 5 \\ -1 & 5 \\ 6 & -6 \end{bmatrix}$$

(3) (c) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-2R_1+R_3 \to R_3]{-2R_1+R_2 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_2 \to R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_2+R_3 \to R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -4 & 1 & 1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{-R_2 \to R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 4 & -1 & -1 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

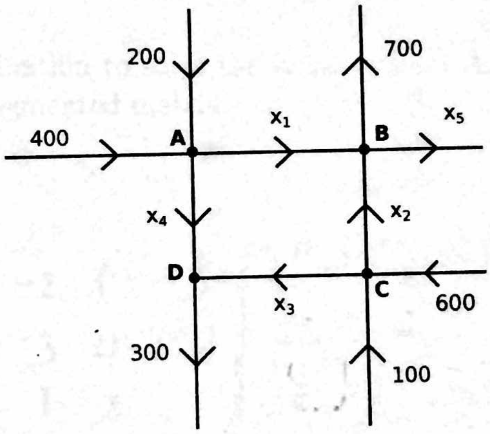
$$\underbrace{\begin{matrix} -1 & 1 & 0 \\ 4 & -1 & -1 \\ -2 & 0 & 1 \end{matrix}}_{A^{-1}}$$

(4) (*a*) Let $T : R^2 \to R^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 + 4x_2)$. Show that T is invertible and find a formula for T^{-1}.

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}^{-1} = \frac{1}{1.4-2.3}\begin{bmatrix}4&-2\\-3&1\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}4&-2\\-3&1\end{bmatrix} = \begin{bmatrix}-2&1\\3/2&-1/2\end{bmatrix}$$

$$T^{-1}(x_1,x_2) = \left(-2x_1+x_2,\ \frac{3}{2}x_1-\frac{1}{2}x_2\right)$$

(*b*) The network in the figure shows the traffic flow (in vehicles per hour) over several one-way streets in the downtown area of a certain city during a typical lunch time. Determine the general flow pattern for the network. In other words, find the general solution of the system of equations that describes the flow. In your general solution let x_4 be free.



$$\begin{aligned} 400+200 &= x_1+x_4 \\ x_1+x_2 &= x_5+700 \\ x_4+x_3 &= 300 \\ 100+600 &= x_2+x_3 \end{aligned} \implies \begin{aligned} x_1+x_4 &= 600 \\ x_1+x_2-x_5 &= 700 \\ x_3+x_4 &= 300 \\ x_2+x_3 &= 700 \end{aligned}$$

$$\begin{bmatrix}1&0&0&1&0&600\\1&1&0&0&-1&700\\0&0&1&1&0&300\\0&1&1&0&0&700\end{bmatrix} \xrightarrow{R_2\leftrightarrow R_1} \begin{bmatrix}1&1&0&0&-1&700\\1&0&0&1&0&600\\0&0&1&1&0&300\\0&1&1&0&0&700\end{bmatrix} \xrightarrow{R_2\leftrightarrow R_4}$$

$$\begin{bmatrix}1&1&0&0&-1&700\\0&1&1&0&0&700\\0&0&1&1&0&300\\1&0&0&1&0&600\end{bmatrix} \xrightarrow{-R_1+R_4\to R_4} \begin{bmatrix}1&1&0&0&-1&700\\0&1&1&0&0&700\\0&0&1&1&0&300\\0&-1&0&1&1&-100\end{bmatrix}$$

$$\xrightarrow{R_2+R_4 \to R_4} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 700 \\ 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & 0 & 1 & 1 & 0 & 300 \\ 0 & 0 & 1 & 1 & 1 & 600 \end{bmatrix} \xrightarrow{-R_3+R_4 \to R_4} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & 700 \\ 0 & 1 & 1 & 0 & 0 & 700 \\ 0 & 0 & 1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 0 & 1 & 300 \end{bmatrix}$$

$\boxed{x_5 = 300}$

$x_3 + x_4 = 300$

$\boxed{x_3 = 300 - x_4.}$

$x_2 + x_3 = 700$

$x_2 = 700 - x_3 = 700 - 300 + x_4$

$\boxed{x_2 = 400 + x_4.}$

$x_1 + x_2 - x_5 = 700$

$x_1 = 700 - x_2 + x_5$

$x_1 = 700 - 400 - x_4 + 300$

$\boxed{x_1 = 600 - x_4.}$