25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 02.04.2011

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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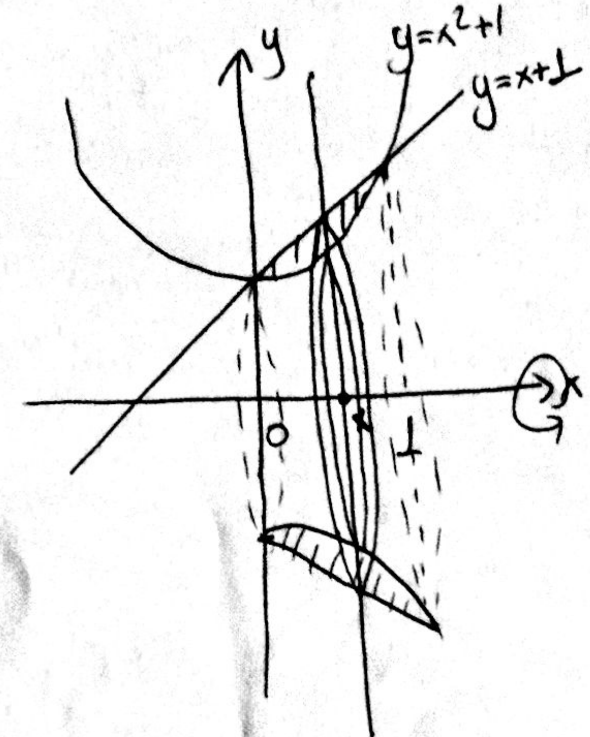
1. Find the volume of the solid obtained by rotating the plane region R bounded by $y = x + 1$ and $y = x^2 + 1$ between $x = 0$ and $x = 1$ about

a) the x-axis using plane slices



$$V = \int_0^1 \pi\left[(x+1)^2 - (x^2+1)^2\right] dx$$

$$= \pi \int_0^1 \left(x^2 + 2x + \cancel{1} - x^4 - 2x^2 - \cancel{1}\right) dx$$

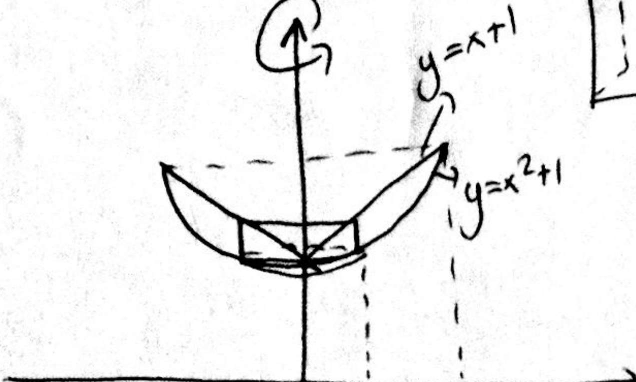
$$= \pi \int_0^1 \left(-x^4 - x^2 + 2x\right) dx$$

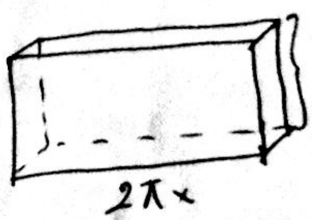
$$= \pi\left(\frac{-x^5}{5} - \frac{x^3}{3} + \frac{\cancel{2}x^2}{\cancel{2}}\right)\Big|_0^1$$

$$= \pi\left(\frac{-1}{5} - \frac{1}{3} + 1\right)$$

$$= \frac{7\pi}{15}$$

b) the y-axis using cylindrical shells





$x+1 - (x^2+1) = x - x^2$

$$V = \int_0^1 2\pi x(x - x^2) dx$$

$$= 2\pi \int_0^1 (x^2 - x^3) dx$$

$$= 2\pi\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1$$

$$= \frac{2\pi}{12} = \frac{\pi}{6}$$

2. Determine whether the given series converges or diverges by using any appropriate test.

a) $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$

$$\int_2^{\infty} \frac{dx}{x(\ln x)^5} = \lim_{R\to\infty} \int_2^{R} \frac{dx}{x(\ln x)^5} \overset{u=\ln x}{=} \lim_{R\to\infty} \int_{\ln 2}^{\ln R} \frac{du}{u^5}$$

$$= \lim_{R\to\infty} -\frac{1}{4u^4}\Big|_{\ln 2}^{\ln R} = -\frac{1}{4}\lim_{R\to\infty}\left[\frac{1}{(\ln R)^4} - \frac{1}{(\ln 2)^4}\right] = \frac{1}{4(\ln 2)^4}$$

By the integral test we get given series converges.

b) $\sum_{n=1}^{\infty} \frac{n^2}{1+n\sqrt{n}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n^2}{1+n\sqrt{n}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^2\sqrt{n}}{1+n\sqrt{n}} = \lim_{n\to\infty} \frac{n^{5/2}}{1+n^{3/2}}$$

$$= \lim_{n\to\infty} \frac{\frac{5}{2}n^{3/2}}{\frac{3}{2}\cdot n^{1/2}} = \infty$$

By the limit comparison test given series diverges to infinity.

c) $\sum_{n=2}^{\infty} \frac{\sqrt{n}}{3^n \ln n}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{\sqrt{n+1}}{3^{n+1}\cdot \ln(n+1)}}{\frac{\sqrt{n}}{3^n \cdot \ln n}} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{3^{n+1}\cdot \ln(n+1)} \cdot \frac{3^n \cdot \ln n}{\sqrt{n}}$$

$$= \frac{1}{3}\lim_{n\to\infty}\sqrt{\frac{n+1}{n}} \cdot \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} = \frac{1}{3}\lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \frac{1}{3}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{3} < 1$$

By the ratio test given series converges.

d) $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n}{n+2}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+2}\right)^{n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{2}{n}\right)^n}$$

$$= \frac{1}{e^2} < 1$$

By the root test $\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2}$ converges.

3. (a) Determine the centre, radius, and the interval of convergence of the given power series $\sum_{n=1}^{\infty} \frac{1}{n}(\frac{x+2}{2})^n$.

$$\sum_{n=1}^{\infty} \frac{1}{2^n \cdot n}(x+2)^n$$

$$R = \lim_{n\to\infty} \frac{2^{n+1}(n+1)}{2^n \cdot n} = 2.$$

Radius of convergence is 2.

The centre of convergence is -2.

$(c-R, c+R) = (-4, 0)$.

For $x=-4$ $\sum_{n=1}^{\infty} \frac{1}{n}(-1)^n$ is an alternating series.

$\lim_{n\to\infty} \frac{1}{n} = 0$, and $|a_{n+1}| < |a_n| \Rightarrow$ converges ✓

For $x=0$ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Thus the interval of convergence $[-4, 0)$.

(b) Find the Taylor series representation of the function $f(x) = ln(2+x)$ in powers of $x-2$.

$$\ln(2+x) = \ln(4+(x-2)) = \ln\left(4\left(1+\frac{x-2}{4}\right)\right)$$

$$= \ln 4 + \ln\left(1+\frac{x-2}{4}\right)$$

$$= \ln 4 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-2)^n}{n4^n}, \quad (-2 < x \leq 6)$$

4. (a) Find the Maclaurin series for the function $J(x) = \int_0^x \frac{e^t - 1}{t} dt$.

$$e^t = \sum_{t=0}^{\infty} \frac{t^n}{n!} \Rightarrow \int_0^x \frac{e^t - 1}{t} dt = \int_0^x \left(1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots\right) dt$$

$$= x + \frac{x^2}{2!\cdot 2} + \frac{x^3}{3!\cdot 3} + \frac{x^4}{4!\cdot 4} + \cdots$$

$$= \sum_{n=1}^{\infty} \frac{x^n}{n!\, n}$$

(b) Evaluate $\lim_{x \to 0} \frac{1 - cos(x^2)}{(1 - cos x)^2}$. $= \lim_{x\to 0} \frac{1 - 1 + \frac{x^4}{2!} - \frac{x^8}{4!} + \cdots}{\left(1 - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots\right)^2}$

$$= \lim_{x\to 0} \frac{x^4\left(\frac{1}{2!} - \frac{x^4}{4!} + \cdots\right)}{x^4\left(\frac{1}{2!} - \frac{x^2}{4!} + \cdots\right)^2}$$

$$= \frac{1/2}{(1/2)^2} = \frac{1}{2} \cdot 4 = 2$$