MATH 154 CALCULUS II

SECOND MİDTERM EXAM
KEY

1. (a) Evaluate the indicated limit or explain why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

(b) Calculate the first partial derivatives of the given function at (0, 0), i.e., $f_x(0,0)$ and $f_y(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{4x^4 + y^4}{x^3 + 3y^3} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Solution

(a) Let $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$.

Find the limit along the line *y* = 0 (the *y*–axis):

$f(x,0) = \dfrac{x^2.0}{x^4 + 0^2} = 0$. Then the value of the limit is 0 as $x \to 0$.

Find the limit along the parabola $y = x^2$:

$f(x,x^2) = \dfrac{x^2.x^2}{x^4 + (x^2)^2} = \dfrac{1}{2}$. Then the value of the limit is 1/2 as $x \to 0$.

Thus, since the value of the limit is different along two different paths, we know that *the limit does not exist*.

(b)

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \Rightarrow f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\dfrac{4h^4 + 0^4}{h^3 + 3.0^3} - 0}{h} = 4$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k) - f(x,y)}{k} \Rightarrow f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{\dfrac{4.0^4 + k^4}{0^3 + 3.k^3} - 0}{h} = \frac{1}{3}$$

2. Given the function $f(x,y)=\dfrac{x+1}{y+1}$.

(a) Find the directional derivative of the function $f(x,y)$ at the point (2, 0) in the direction of the vector $\mathbf{i}+\sqrt{3}\mathbf{j}$.

(b) Find the equations of the tangent plane and normal line to the graph of the function $f(x,y)$ at the point (2, 0).

Solution

(a) $$f_x(x,y)=\frac{1}{y+1},\ f_y(x,y)=-\frac{x+1}{(y+1)^2} \quad \Rightarrow \quad f_x(2,0)=1,\ f_y(2,0)=-3$$

$$\nabla f(2,0)=f_x(2,0)\mathbf{i}+f_y(2,0)\mathbf{j}=\mathbf{i}-3\mathbf{j}$$

$$\mathbf{u}=\frac{\mathbf{i}+\sqrt{3}\mathbf{j}}{\sqrt{1^2+(\sqrt{3})^2}}=\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}$$

$$D_{\mathbf{u}}f(2,0)=\nabla f(2,0)\bullet\mathbf{u}=\left[\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}\right]\bullet\left(\mathbf{i}-3\mathbf{j}\right)=\frac{1}{2}.1+\frac{\sqrt{3}}{2}.(-3)=\frac{1-3\sqrt{3}}{2}$$

(b) The equation of the *tangent plane*:

$$z=f(a,b)+f_x(a,b).(x-a)+f_y(a,b).(y-b)$$

$$z=f(2,0)+f_x(2,0).(x-2)+f_y(0,0).(y-0)=3+1.(x-2)+(-3).(y-0)=x-3y+3$$

The equation of the *normal line*:

$$\frac{x-a}{f_x(a,b)}=\frac{y-b}{f_y(a,b)}=\frac{z-f(a,b)}{-1} \quad \Rightarrow \quad \frac{x-2}{f_x(2,0)}=\frac{y-0}{f_y(2,0)}=\frac{z-f(2,0)}{-1}$$

$$\Rightarrow \quad \frac{x-2}{1}=\frac{y-0}{-3}=\frac{z-3}{-1}$$

3. (a) Find and classify the critical points of the given function $f(x,y)=x^4+2y^2-4xy$.

(b) Find the maximum and minimum of the function $f(x,y)=x+6y-7$ on $x^2+3y^2=13$ by using the method of Lagrange multipliers.

Solution

(a)

$$f(x,y)=x^4+2y^2-4xy$$
$$f_x(x,y)=4x^3-4y,\quad f_y(x,y)=4y-4x,\quad f_{xx}(x,y)=12x^2,\quad f_{yy}(x,y)=4,\quad f_{xy}(x,y)=-4$$

For critical points:

$$\left.\begin{array}{l} f_x(x,y)=4x^3-4y=0\Rightarrow y=x^3 \\ f_y(x,y)=4y-4x=0\ \Rightarrow y=x \end{array}\right\}\Rightarrow x^3=x\Rightarrow x^3-x=0\Rightarrow x=0,x=1,x=-1$$
$$\Rightarrow (0,0),(1,1),(-1,-1)$$

At (0, 0):

$$A=f_{xx}(0,0)=0,\ B=f_{xy}(0,0)=-4,\ C=f_{yy}(0,0)=4\Rightarrow B^2-AC=16>0\Rightarrow \text{saddle point}$$

At (1, 1):

$$A=f_{xx}(1,1)=12,\ B=f_{xy}(1,1)=-4,\ C=f_{yy}(1,1)=4\Rightarrow B^2-AC=-32<0, A<0\Rightarrow \text{minimum}$$

At (1, 1):

$$A=f_{xx}(-1,-1)=12,\ B=f_{xy}(-1,-1)=-4,\ C=f_{yy}(-1,-1)=4\Rightarrow B^2-AC=-32<0, A<0$$
$$\Rightarrow \text{minimum}$$

(b)

$$L(x,y,\lambda)=f(x,y)+\lambda g(x,y)=x+6y-7+\lambda\left(x^2+3y^2-13\right)$$

$$\left.\begin{array}{l} \dfrac{\partial L}{\partial x}=1+2\lambda x=0\Rightarrow x=-\dfrac{1}{2\lambda} \\ \dfrac{\partial L}{\partial y}=6+6\lambda y=0\Rightarrow y=-\dfrac{1}{\lambda} \\ \dfrac{\partial L}{\partial \lambda}=x^2+3y^2-13=0 \end{array}\right\}\Rightarrow x^2+3y^2-13=\left(-\frac{1}{2\lambda}\right)^2+3\left(-\frac{1}{\lambda}\right)^2-13=0\Rightarrow \lambda=\pm\frac{1}{2}$$

When λ = –1/2 we get x = 1 and y = 2. When λ = 1/2 we get x = –1 and y = –2. The points are (1, 2) and (–1, –2). Then f(1, 2) = 6 (*maximum*) and f(–1, –2) = –20 (*minimum*).

4. (a) Sketch the domain of integration and evaluate the given iterated integral $\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx$.

(b) Evaluate the double integral $\iint_R (3x^2 + 3y^2)\,dA$, where R is the region given by $x \ge 0, y \ge 0$ and $1 \le x^2 + y^2 \le 4$.

Solution

(a)

$$\int_0^4 \int_{\sqrt{x}}^2 \sin(y^3)\,dy\,dx = \int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 \sin(y^3)\,x\Big|_0^{y^2} dy$$

$$= \int_0^2 y^2 \sin(y^3)\,dy \qquad \text{Let } y^3 = u.$$

$$= \int_0^8 \sin(u)\frac{du}{3} = \frac{-1}{3}\cos u\Big|_0^8 = \frac{-1}{3}(\cos 8 - 1)$$

$$x = r\cos\theta, y = r\sin\theta \Rightarrow x^2 + y^2 = r^2 \qquad dA = r\,dr\,d\theta$$

(b) Polar coordinates: $x^2 + y^2 = r^2 = 1 \Rightarrow r = 1, x^2 + y^2 = r^2 = 4 \Rightarrow r = 2$ so $1 \le r \le 2$

in the first quadrant $\Rightarrow 0 \le \theta \le \frac{\pi}{2}$

$$\iint_R 3(x^2 + y^2)\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 3r^2 r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \frac{3}{4} r^4\Big|_1^2 d\theta = \frac{45}{4}\int_0^{\frac{\pi}{2}} d\theta = \frac{45\pi}{8}$$