KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 18.05.2013

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

Name: ....

Student No: ....

Department: ....

Section: **Check for your instructor below:**

- ☐ Tahsin Öner
- ☐ Sevin Gümgüm
- ☐ Burak Ordin
- ☐ Güvenç Arslan

1. (a) Evaluate the indicated limit or explain why it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y^2}{x^4+3y^4} = \frac{0}{0}$$

Find the limit along the x-axis (y=0)!

$f(x,0) = 0$

Find the limit along the y-axis (x=0)!

$f(0,y) = 0$

Find the limit along y=x:

$f(x,x) = \frac{2x^2 \cdot x^2}{x^4+3x^4} = \frac{2x^4}{4x^4} = \frac{1}{2}$

So, the limit does not exist.

(b) Calculate $f_x(0,0)$ and $f_y(0,0)$ of the given function

$$f(x,y) = \begin{cases} \dfrac{\sin(x^3-y^3)}{2x^2+3y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin(h^3)}{2h^2}}{h} = \lim_{h\to 0} \frac{\sin(h^3)}{2h^3} = \frac{1}{2}$

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{\sin(-k^3)}{3k^2}}{k} = \lim_{k\to 0} \frac{-\sin(k^3)}{3k^3}$

$= \frac{-1}{3}$

2. (a) Find the rate of change of the function $f(x) = x\cos y$ at the point $(0, \pi)$ in the direction of the vector $u = 2i + j$.

$f_x(x,y) = \cos y \Rightarrow f_x(0,\pi) = \cos\pi = -1$

$f_y(x,y) = -x\sin y \Rightarrow f_y(0,\pi) = 0$

$\nabla f(0,\pi) = -1 \cdot \vec{i} + 0 \cdot \vec{j} = -i$

$u = 2\vec{i} + \vec{j}$ is not unit, so divide by $|u| = \sqrt{5}$

$\frac{u}{|u|} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$

$D_{\frac{u}{|u|}} f(0,\pi) = \left(\frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}\right) \cdot (-\vec{i}) = \frac{-2}{\sqrt{5}}$

(b) Find the equations of the tangent plane and the normal line to the graph of the function $f(x) = 32 - 3x^2 - 4y^2$ at the point where $x = 2$ and $y = 1$.

$f_x(x,y) = -6x \Rightarrow f_x(2,1) = -12$

$f_y(x,y) = -8y \Rightarrow f_y(2,1) = -8$

$f(2,1) = 16$

Tangent Plane : $z = 16 - 12(x-2) - 8(y-1)$

Normal Line : $\frac{x-2}{-12} = \frac{y-1}{-8} = \frac{z-16}{-1}$

3. (a) Use Lagrange Multipliers to find the maximum and the minimum values of $f(x, y) = x + 2y$ subject to the constraint $x^2 + y^2 = 1$.

$$L(x,y,\lambda) = x + 2y + \lambda(x^2 + y^2 - 1)$$

$$\left.\begin{aligned}\frac{\partial L}{\partial x} &= 1 + 2\lambda x = 0 \Rightarrow \lambda = \frac{-1}{2x} \\ \frac{\partial L}{\partial y} &= 2 + 2\lambda y = 0 \Rightarrow \lambda = \frac{-1}{y}\end{aligned}\right\} \Rightarrow \frac{-1}{2x} = \frac{-1}{y} \Rightarrow y = 2x$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 1 = 0 \Rightarrow x^2 + 4x^2 = 1 \Rightarrow x = \mp\frac{1}{\sqrt{5}} \Rightarrow y = \mp\frac{2}{\sqrt{5}}$$

$f\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right) = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}}$ is the abs. max value

$f\left(\frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}}\right) = \frac{-1}{\sqrt{5}} - \frac{4}{\sqrt{5}} = \frac{-5}{\sqrt{5}}$ is the abs. min value.

(b) Find and classify all the critical points of the function $f(x, y) = xy(x-2)(y+3)$.

$$f_x(x,y) = y(x-2)(y+3) + xy(y+3) = y(y+3)(2x-2) = 0 \Rightarrow y=0, y=-3, x=1$$

$$f_y(x,y) = x(x-2)(y+3) + x(x-2)\cdot y = (x-2)\cdot x(2y+3) = 0 \Rightarrow x=0, x=2, y=\frac{-3}{2}$$

$$f_{xx}(x,y) = 2y(y+3), \quad f_{xy}(x,y) = (2x-2)(2y+3), \quad f_{yy}(x,y) = 2x(x-2)$$

$y=0 \Rightarrow f_x = 0,\ f_y = x(x-2)\cdot 3 = 0 \Rightarrow x=0, x=2$

$y=-3 \Rightarrow f_x = 0,\ f_y = x(x-2)(-3) = 0 \Rightarrow x=0, x=2$

$x=1 \Rightarrow f_x = 0,\ f_y = (-1)\cdot 1\cdot(2y+3) = 0 \Rightarrow y = -3/2$

$x=0 \Rightarrow f_y = 0,\ f_x = y(y+3)(-2) = 0 \Rightarrow y=0, y=-3$

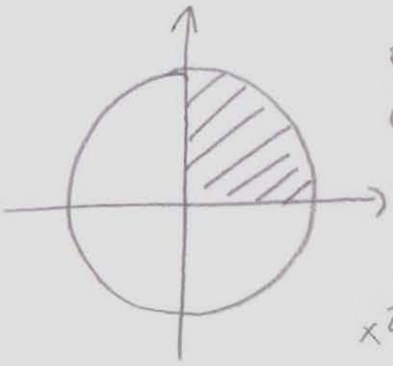
$x=2 \Rightarrow f_y = 0,\ f_x = y(y+3)(2) = 0 \Rightarrow y=0, y=-3$

$y=\frac{-3}{2} \Rightarrow f_y = 0,\ f_x = 0 \Rightarrow x=1$

$(0,0), (2,0), (0,-3), (2,-3), (1,-3/2)$ are the critical points.

$(0,0)$:	$(2,0)$:	$(0,-3)$:	$(2,-3)$:	$(1,-3/2)$:
$A = f_{xx}(0,0) = 0$	$A = f_{xx}(2,0) = 0$	$A = f_{xx}(0,-3) = 0$	$A = f_{xx}(2,-3) = 0$	$A = f_{xx}(1,\frac{-3}{2}) = \frac{-9}{2}$
$B = f_{xy}(0,0) = -6$	$B = f_{xy}(2,0) = 6$	$B = f_{xy}(0,-3) = 6$	$B = f_{xy}(2,-3) = -6$	$B = f_{xy}(1,\frac{-3}{2}) = 0$
$C = f_{yy}(0,0) = 0$	$C = f_{yy}(2,0) = 0$	$C = f_{yy}(0,-3) = 0$	$C = f_{yy}(2,-3) = 0$	$C = f_{yy}(1,\frac{-3}{2}) = -1$
$B^2 - AC > 0$	$B^2 - AC > 0$	$B^2 - AC > 0$	$B^2 - AC > 0$	$B^2 - AC < 0$, $A < 0$
saddle point	saddle point	saddle point	saddle point	local max point

4. (a) Evaluate the integral $\int_D\int \cos(x^2+y^2)\, dA$ where D is the region given by $x \geq 0$, $y \geq 0$ and $x^2+y^2 \leq 4$



$0 \leq r \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$$\int_0^2\int_0^{\pi/2} \cos(r^2)\cdot r\, d\theta\, dr$$

$$= \int_0^2 r\cdot\cos(r^2)\cdot\left(\theta\Big|_0^{\frac{\pi}{2}}\right) dr$$

$$= \frac{\pi}{2}\int_0^2 r\cos(r^2)\, dr$$

Let $u = r^2$

$du = 2r\,dr$

$\frac{1}{2}du = r\,dr$

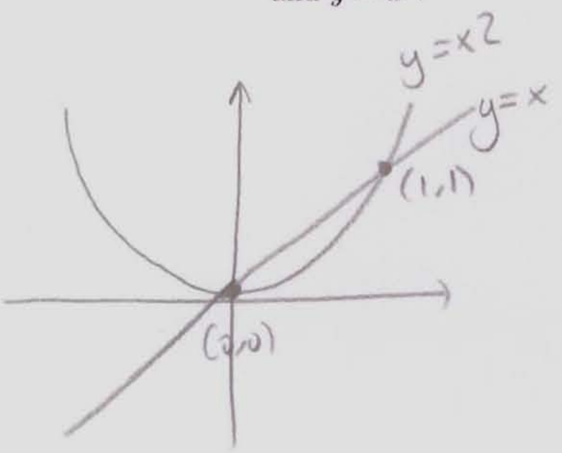
$r=0 \Rightarrow u=0$

$r=2 \Rightarrow u=4$

$$= \frac{\pi}{2}\int_0^4 \cos u \cdot \frac{1}{2}\, du$$

$$= \frac{\pi}{4} \sin u\Big|_0^4 = \frac{\pi}{4}\sin 4$$

(b) Evaluate the integral $\int_R\int (x+y^2)\, dA$ where R is the region bounded by $y=x$ and $y=x^2$.



$$\int_0^1\int_{x^2}^{x} (x+y^2)\, dy\, dx$$

$$= \int_0^1 \left(xy + \frac{y^3}{3}\right)\Big|_{x^2}^{x} dx$$

$$= \int_0^1 \left[\left(x^2+\frac{x^3}{3}\right) - \left(x^3+\frac{x^6}{3}\right)\right] dx$$

$$= \int_0^1 \left(x^2 - \frac{2x^3}{3} - \frac{x^6}{3}\right) dx$$

$$= \left(\frac{x^3}{3} - \frac{2x^4}{12} - \frac{x^7}{21}\right)\Big|_0^1 = \frac{5}{42}$$