KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS I 12.04.2013

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. Evaluate the given improper integral or show that it diverges

(a) $\int_0^\infty \frac{2x}{(x^2+1)^5}dx = \lim_{R\to\infty} \int_0^R \frac{2x}{(x^2+1)^5} \cdot dx$

Let $u = x^2+1$
$du = 2x\,dx$

$= \lim_{R\to\infty} \int \frac{1}{u^5} \cdot du$

$= \lim_{R\to\infty} \left(\frac{u^{-4}}{-4} \right)$

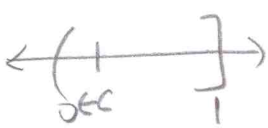
$= \lim_{R\to\infty} \left(\frac{-1}{4} \frac{1}{(x^2+1)^4} \right) \Big|_0^R$

$= \lim_{R\to\infty} \left(\frac{-1}{4} \left(\frac{1}{(R^2+1)^4} - 1 \right) \right)$

$= \frac{1}{4}$

So, the integral converges to $\frac{1}{4}$

(b) $\int_0^1 \frac{(\ln x)^2}{x} dx = \lim_{c\to 0^+} \int_c^1 \frac{(\ln x)^2}{x} dx$



Let $u = \ln x$
$du = \frac{1}{x} dx$

$= \lim_{c\to 0^+} \int u^2 du$

$= \lim_{c\to 0^+} \left(\frac{u^3}{3} \right)$

$= \lim_{c\to 0^+} \left(\frac{(\ln x)^3}{3} \right) \Big|_c^1$

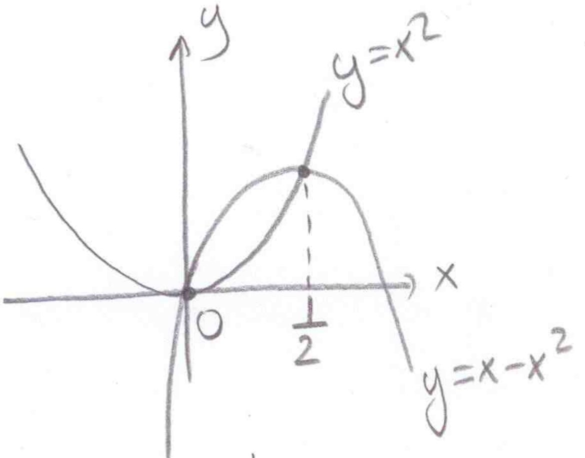
$= \lim_{c\to 0^+} \left(\frac{(\ln 1)^3}{3} - \frac{(\ln c)^3}{3} \right)$

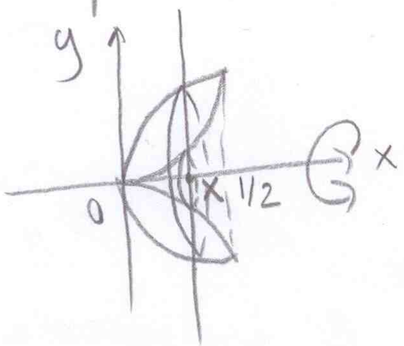
$= \infty$

So, the integral diverges to ∞.

2. Sketch the graph of the region bounded by the functions $y = x - x^2$ and $y = x^2$. Then, find the volume of the solid generated by revolving this region about

(a) the x-axis using the slicing method.





$$V=\int_0^{1/2} \pi\left[(x-x^2)^2-(x^2)^2\right]\cdot dx$$

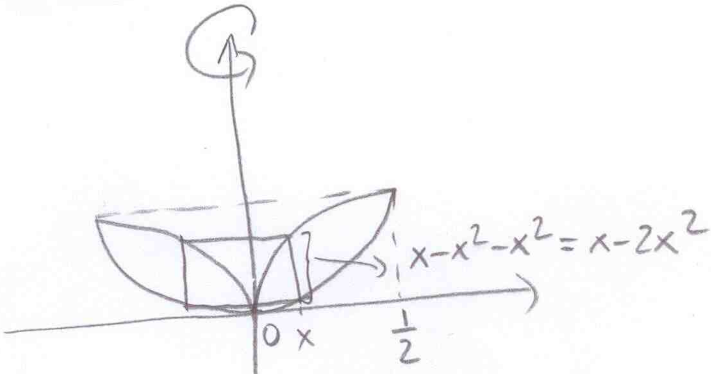
$$=\pi\int_0^{1/2}\left(x^2-2x^3\right)dx$$

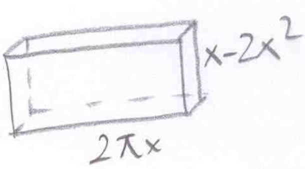
$$=\pi\left(\frac{x^3}{3}-\frac{2x^4}{4}\right)\Bigg|_0^{1/2}$$

$$=\pi\left(\frac{\frac{1}{8}}{3}-\frac{\frac{1}{16}}{2}\right)$$

$$=\frac{\pi}{96}$$

(b) the y-axis using the cylindrical shells.





$$V=\int_0^{1/2} 2\pi x\left(x-2x^2\right)dx$$

$$=2\pi\int_0^{1/2}\left(x^2-2x^3\right)dx$$

$$=2\pi\left(\frac{x^3}{3}-\frac{2x^4}{4}\right)\Bigg|_0^{1/2}$$

$$=\frac{\pi}{48}$$

3. Determine whether the given series converge or diverge by applying an appropriate convergence test.

(a) $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

Use comparison test:

$\cos n \leq 1$

$\frac{\cos n}{n^3} \leq \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{\cos n}{n^3} \leq \sum_{n=1}^{\infty} \frac{1}{n^3}$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by p-series ($p=3>1$). So, $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$ also converges.

(b) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Use Ratio Test:

$a_n = \frac{2^n}{n!}$

$a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}$

$= \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2}{\cancel{2^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{n!}} = 0$

So, the series converges.

(c) $\sum_{n=1}^{\infty} \frac{1}{2n - \sqrt{n}}$

Use Limit Comparison Test:

Compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty} \frac{\frac{1}{2n-\sqrt{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2n-\sqrt{n}} = \lim_{n\to\infty} \frac{1}{2-\underbrace{\frac{1}{\sqrt{n}}}_{0}} = \frac{1}{2}$

Since the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{2n-\sqrt{n}}$ also diverges.

4. Using the power series representation of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots \qquad ; |x| < 1$$

find the power series representation for the given functions in powers of $x-2$. State the intervals where each series is valid.

(a) $\dfrac{1}{x+3}$

$$\frac{1}{x+3} = \frac{1}{x-2+2+3} = \frac{1}{5+(x-2)} = \frac{1}{5\left[1-\left(-\frac{(x-2)}{5}\right)\right]}$$

$$= \frac{1}{5} \cdot \sum_{n=0}^{\infty} \left(-\frac{(x-2)}{5}\right)^n \; ; \; \left|-\frac{(x-2)}{5}\right| < 1$$

$$\frac{1}{x+3} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} \cdot (x-2)^n \; ; \; -3 < x < 7$$

(b) $\dfrac{1}{(x+3)^2}$

$$\left(\frac{1}{x+3}\right)' = \frac{-1}{(x+3)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{5^{n+1}} \cdot n(x-2)^{n-1} \; ; \; -3 < x < 7$$

$$\frac{1}{(x+3)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^{n+1}} \cdot n \cdot (x-2)^{n-1} \; ; \; -3 < x < 7$$