25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 26.04.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

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1. (a) Starting with the power series representation

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad , \quad -1 < x < 1$$

determine the power series representation for the function $\frac{1}{x^2}$ in powers of $x-2$. On what interval is the representation valid?

First, find power series representation of $\frac{1}{x}$:

Let $t = x-2$, $x = 2+t$

$$\frac{1}{x} = \frac{1}{2+t} = \frac{1}{2(1+\frac{t}{2})} = \frac{1}{2}\cdot\frac{1}{1-(-\frac{t}{2})} = \frac{1}{2}\cdot\sum_{k=0}^{\infty}\left(-\frac{t}{2}\right)^k ; \left|-\frac{t}{2}\right|<1$$

$$\therefore \frac{1}{x} = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{(x-2)^k}{2^{k+1}} ; \left|\frac{x-2}{2}\right|<1 \Rightarrow -2<x-2<2$$

Differentiate both sides:

$$\frac{-1}{x^2} = \sum_{k=1}^{\infty}(-1)^k \cdot \frac{k\cdot(x-2)^{k-1}}{2^{k+1}} \Rightarrow \frac{1}{x^2} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}\cdot k}{2^{k+1}}\cdot(x-2)^{k-1}$$

the representation is valid on $0<x<4$

(b) Find the Maclaurin series for the function $L(x) = \int_0^x \sin(t^2)\,dt$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{2k+1}}{(2k+1)!}$$

$$\sin(t^2) = t^2 - \frac{(t^2)^3}{3!} + \frac{(t^2)^5}{5!} - \frac{(t^2)^7}{7!} + \cdots$$

$$L(x) = \int_0^x \sin(t^2)\,dt = \int_0^x \left[t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - \frac{t^{14}}{7!} + \cdots\right]dt$$

$$= \left(\frac{t^3}{3} - \frac{t^7}{7\cdot 3!} + \frac{t^{11}}{11\cdot 5!} - \frac{t^{15}}{15\cdot 7!} + \cdots\right)\Bigg|_0^x$$

$$= \frac{x^3}{3} - \frac{x^7}{7\cdot 3!} + \frac{x^{11}}{11\cdot 5!} - \frac{x^{15}}{15\cdot 7!} + \cdots$$

$$= \sum_{k=0}^{\infty}(-1)^k \cdot \frac{x^{4k+3}}{(4k+3)\cdot(2k+1)!}$$

2. (a) Evaluate the indicated limit or explain why it does not exist

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2} \quad \left(\frac{0}{0}\right)$$

Find the limit along the x-axis $(y=0)$!

$$f(x,0) = \frac{(x+0)^2}{x^2+(0)^2} = \frac{x^2}{x^2} = 1$$

Find the limit along the y-axis $(x=0)$:

$$f(0,y) = \frac{(0+y)^2}{(0)^2+y^2} = \frac{y^2}{y^2} = 1$$

Find the limit along $y=x$:

$$f(x,x) = \frac{(x+x)^2}{x^2+x^2} = \frac{4x^2}{2x^2} = 2$$

Since the limits are different, the limit does not exist.

(b) Calculate the first partial derivatives of the given function at $(0,0)$, that is $f_x(0,0)$ and $f_y(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{3x^3-2y^3}{x^2+y^2} & ; \ (x,y) \neq (0,0) \\ 0 \ ; \ (x,y) = (0,0) \end{cases}$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{3h^3-0}{h^2+0} - 0}{h} = 3$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k)-f(0,0)}{k} = \lim_{k\to 0} \frac{\frac{0-2k^3}{0+k^2} - 0}{k} = -2$$

3. (a) Find the directions in which the directional derivative of $f(x,y) = ye^{-xy}$ at $(0,2)$ have rate of change 1?

Let $\vec{u} = a\vec{i} + b\vec{j}$ be the unit direction vector.

we have $\sqrt{a^2+b^2} = 1 \Rightarrow a^2+b^2 = 1$

$f_x(x,y) = -y^2 e^{-xy} \Rightarrow f_x(0,2) = -4$

$f_y(x,y) = e^{-xy} - xye^{-xy} \Rightarrow f_y(0,2) = 1$

$\nabla f(0,2) = f_x(0,2)\vec{i} + f_y(0,2)\vec{j} = -4\vec{i} + \vec{j}$

$D_u f(0,2) = (a\vec{i} + b\vec{j}) \cdot (-4\vec{i} + \vec{j}) = -4a + b = 1$

$\Rightarrow b = 1 + 4a$ and $a^2 + b^2 = 1$, then we have

$a^2 + (1+4a)^2 = 1 \Rightarrow 17a^2 + 8a + \cancel{1} = \cancel{1} \Rightarrow a = 0, a = \frac{-8}{17}$

$a = 0 \Rightarrow b = 1 \Rightarrow \boxed{u = \vec{j}}$ & $a = \frac{+8}{17} \Rightarrow b = \frac{-15}{17} \Rightarrow \boxed{u = \frac{-8}{17}\vec{i} - \frac{15}{17}\vec{j}}$

(b) Find equations of the tangent plane and the normal line to the graph of the function $f(x,y) = \ln\left(1 + \frac{y}{x}\right)$ at $(1,1)$.

$f_x(x,y) = \frac{1}{1+\frac{y}{x}} \cdot \frac{-y}{x^2} \Rightarrow f_x(1,1) = \frac{-1}{2}$

$f_y(x,y) = \frac{1}{1+\frac{y}{x}} \cdot \frac{1}{x} \Rightarrow f_y(1,1) = \frac{1}{2}$

$f(1,1) = \ln(2)$

Tangent Plane: $z = \ln 2 - \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$

Normal Line: $\frac{x-1}{-1/2} = \frac{y-1}{1/2} = \frac{z - \ln 2}{-1}$

4. Find and classify the critical points of the given function $f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$.

$f_x(x,y) = 3x^2 + 3y^2 - 15 = 0 \Rightarrow x^2 + y^2 = 5$

$f_y(x,y) = 6xy + 3y^2 - 15 = 0 \Rightarrow 2xy + y^2 = 5$

$x^2 + y^2 = 5 \Rightarrow y^2 = 5 - x^2$

$2xy + y^2 = 5 \Rightarrow 2xy + \cancel{5} - x^2 = \cancel{5}$

$x(2y - x) = 0 \Rightarrow x = 0,\ x = 2y$

Critical points

$x = 0 \Rightarrow y = \mp\sqrt{5}$

$x = 2y \Rightarrow 4y^2 + y^2 = 5 \Rightarrow y = \mp 1 \Rightarrow x = \mp 2$

$(0,\sqrt{5}), (0,-\sqrt{5}), (2,1), (-2,-1)$

$f_{xx}(x,y) = 6x,\ f_{xy}(x,y) = 6y,\ f_{yy}(x,y) = 6x + 6y$

For $(0,\sqrt{5})$:

$A = f_{xx}(0,\sqrt{5}) = 0$

$B = f_{xy}(0,\sqrt{5}) = 6\sqrt{5}$

$C = f_{yy}(0,\sqrt{5}) = 6\sqrt{5}$

$B^2 - AC > 0$

saddle point

For $(0,-\sqrt{5})$:

$A = f_{xx}(0,-\sqrt{5}) = 0$

$B = f_{xy}(0,-\sqrt{5}) = -6\sqrt{5}$

$C = f_{yy}(0,-\sqrt{5}) = -6\sqrt{5}$

$B^2 - AC$

saddle point

For $(2,1)$:

$A = f_{xx}(2,1) = 12$

$B = f_{xy}(2,1) = 6$

$C = f_{yy}(2,1) = 18$

$B^2 - AC < 0$

$A > 0$

local min point

For $(-2,-1)$:

$A = f_{xx}(-2,-1) = -12$

$B = f_{xy}(-2,-1) = -6$

$C = f_{yy}(-2,-1) = -18$

$B^2 - AC < 0$

$A < 0$

local max point