25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 20.03.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. (a) Evaluate the given integral or show that it diverges: $\int_{-\infty}^{\infty} 2xe^{-2x^2}\,dx$

$$\int_{-\infty}^{\infty} 2x \cdot e^{-2x^2} dx = \int_{-\infty}^{0} 2xe^{-2x^2} dx + \int_{0}^{\infty} 2xe^{-2x^2} dx$$

$$= \lim_{t \to -\infty} \int_{t}^{0} 2xe^{-2x^2} dx + \lim_{R \to \infty} \int_{0}^{R} 2xe^{-2x^2} dx$$

Let $u = -2x^2$ then $du = -4x\,dx$, $-\frac{1}{2} du = 2x\,dx$

$$= \lim_{t \to -\infty} \int e^{u} \cdot \left(-\tfrac{1}{2}\right) \cdot du + \lim_{R \to \infty} \int e^{u} \cdot \left(-\tfrac{1}{2}\right) \cdot du$$

$$= -\frac{1}{2} \lim_{t \to -\infty} e^{-2x^2} \Big|_{t}^{0} - \frac{1}{2} \lim_{R \to \infty} e^{-2x^2} \Big|_{0}^{R}$$

$$= -\frac{1}{2} \lim_{t \to -\infty} \left(1 - \underbrace{e^{-2t^2}}_{\to 0}\right) - \frac{1}{2} \lim_{R \to \infty} \left(\underbrace{e^{-2R^2}}_{\to 0} - 1\right) = 0$$

converges to 0.

(b) Find the length of the curve $4y = 2 \ln x - x^2$ from $x = 1$ to $x = e$.

$$L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \cdot dx$$

$$y = f(x) = \frac{1}{2} \ln x - \frac{1}{4} x^2$$

$$f'(x) = \frac{1}{2x} - \frac{x}{2} \Rightarrow (f'(x))^2 = \left(\frac{1}{2x}\right)^2 - \frac{1}{2} + \left(\frac{x}{2}\right)^2$$

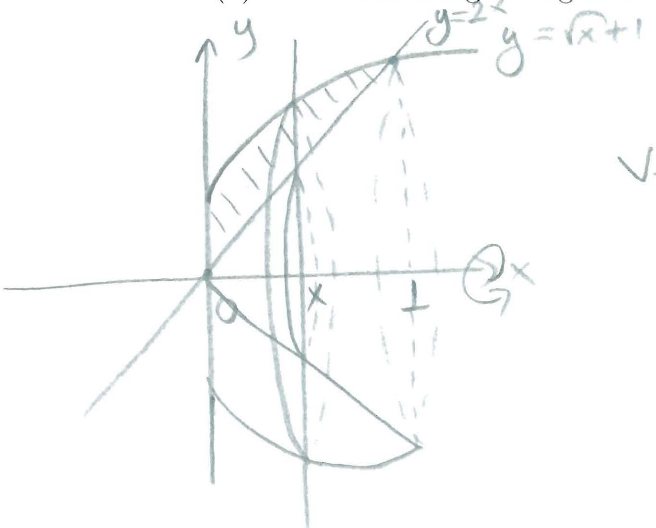
$$1 + (f'(x))^2 = \left(\frac{1}{2x}\right)^2 + \frac{1}{2} + \left(\frac{x}{2}\right)^2 = \left(\frac{1}{2x} + \frac{x}{2}\right)^2$$

$$L = \int_{1}^{e} \sqrt{\left(\frac{1}{2x} + \frac{x}{2}\right)^2}\, dx = \int_{1}^{e} \left(\frac{1}{2x} + \frac{x}{2}\right) dx$$

$$= \left(\frac{1}{2} \ln|x| + \frac{x^2}{4}\right)\Big|_{1}^{e} = \left(\frac{1}{2} \underbrace{\ln e}_{=1} + \frac{e^2}{4}\right) - \left(\frac{1}{2} \underbrace{\ln 1}_{=0} + \frac{1}{4}\right)$$

2. Sketch the graph of the region bounded by $y = \sqrt{x} + 1$, $y = 2x$, $x = 0$, $x = 1$ and y-axis. Then, find the volume of the solid generated by rotating this region about

(a) the x-axis using slicing method.



$$V = \int_0^1 \left[\pi \cdot (\sqrt{x}+1)^2 - \pi \cdot (2x)^2\right] dx$$

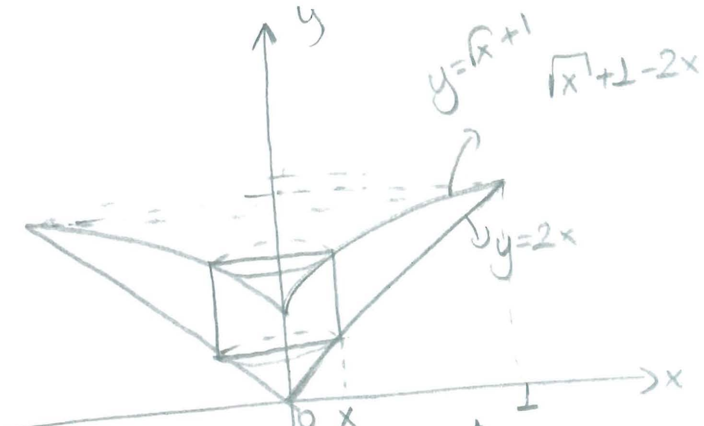
$$= \pi \int_0^1 \left[x + 2\sqrt{x} + 1 - 4x^2\right] dx$$

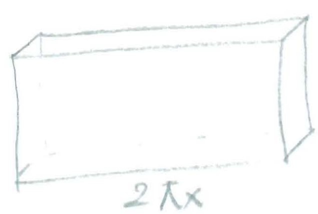
$$= \pi \left(\frac{x^2}{2} + 2 \cdot \frac{x^{3/2}}{3/2} + x - \frac{4x^3}{3}\right)\Bigg|_0^1$$

$$= \pi \left(\frac{1}{2} + \frac{4}{3} + 1 - \frac{4}{3}\right)$$

$$= \frac{3\pi}{2} \text{ cubed units}$$

(b) the y-axis using cylindrical shell method.





$$V = \int_0^1 2\pi x \left(\sqrt{x} + 1 - 2x\right) dx$$

$$= 2\pi \int_0^1 \left(x^{3/2} + x - 2x^2\right) dx$$

$$= 2\pi \left(\frac{x^{5/2}}{5/2} + \frac{x^2}{2} - \frac{2x^3}{3}\right)\Bigg|_0^1$$

7π cubed units

3. (a) Determine whether the given series converges or diverges by using any appropriate test.

i. $\sum_{n=1}^{\infty} \frac{n+2^n}{1+3^n}$ $= \sum_{n=1}^{\infty} \frac{n}{1+3^n} + \sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges

$\underbrace{\sum_{n=1}^{\infty} \frac{n}{1+3^n}}_{\text{converges by comparison test}} < \underbrace{\sum_{n=1}^{\infty} \frac{n}{3^n}}_{\text{converges by ratio test}}$

$\lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{1}{3} < 1$

$\underbrace{\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}}_{\text{converges by comparison test}} < \underbrace{\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n}_{\text{converges by geometric series}}$

ii. $\sum_{n=2}^{\infty} \frac{n^2 \cos(n\pi)}{n^4-1}$ So, this series converges absolutely.

Since the series is alternating we first look for the absolute convergence

$\sum_{n=2}^{\infty} \left| \frac{n^2 \cdot \cos(n\pi)}{n^4-1} \right| = \sum_{n=2}^{\infty} \frac{n^2}{n^4-1}$ compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{\frac{n^2}{n^4-1}}{\frac{1}{n^2}} = 1$. So, by limit comparison test the convergence of $\sum \frac{1}{n^2}$ (by p-series) implies the convergence of $\sum \frac{n^2}{n^4-1}$

(b) Find the sum of the given series: $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \sum_{n=1}^{\infty} \left(\frac{1/4}{4n-3} - \frac{1/4}{4n+1} \right) = \frac{1}{4} \sum_{n=1}^{\infty} \left(\frac{1}{4n-3} - \frac{1}{4n+1} \right)$

n-th partial sum of the above series is:

$S_n = \frac{1}{4}\left[1 - \cancel{\frac{1}{5}} + \frac{1}{5} - \frac{1}{9} + \frac{1}{9} - \frac{1}{13} + \cdots + \frac{1}{4n-3} - \frac{1}{4n+1} \right]$

$S_n = \frac{1}{4}\left[1 - \frac{1}{4n-1} \right]$

$\lim \frac{1}{4}\left[1 - \frac{1}{\quad} \right] = \frac{1}{4}$ So $\sum_{n=1}^{\infty} \frac{1}{(4n-3)(4n+1)} = \frac{1}{4}$

4. Find the center, radius and interval of convergence of the given power series: $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n \sqrt{n}}$

$x=2$ is the center of convergence

$a_n = \frac{1}{3^n \cdot \sqrt{n}}$ so $a_{n+1} = \frac{1}{3^{n+1} \sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{3^{n+1}\sqrt{n+1}}}{\frac{1}{3^n \cdot \sqrt{n}}} = \frac{1}{3}$$

Radius of convergence: $R = \frac{1}{1/3} = 3$

Interval of convergence: 

At $x=-1$: $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges conditionally

$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series

So, the series does not converge absolutely.

Use alternating series test:

i) $a_n \cdot a_{n+1} < 0$ ✓

ii) $|a_{n+1}| \leq |a_n|$ ✓ $|a_{n+1}| = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = |a_n|$

iii) $\lim_{n\to\infty} a_n = 0$ ✓ $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

At $x=5$: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by p-series

Interval of convergence: $[-1, 5)$