SOLUTIONS

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 07.05.2011

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

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1. a) Show that

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

is continuous at every point except the origin.

$f(x,y)$ is continuous at (a,b) iff $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

On the line $y=0$: $\lim f(x,y) = 0$

On the line $x=0$: $\lim f(x,y) = 0$

On the line $y=x$: $\lim\limits_{x\to 0} \dfrac{2x^2}{x^2+x^2} = 1$

$\Rightarrow$ So, the limit does not exist at $(0,0)$. f is discontinuous at $(0,0)$.

For all other points:

$f(x,y)$ is defined and continuous since $f(x,y)$ is a rational functor with a numerator of degree 2 and denumerator of degree 2. The only points which make it discontinuous are the roots of denumerator $(0,0)$.

$$\text{b) } f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ exist but f is not differentiable at $(0,0)$.

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} \Rightarrow f_x(0,0) = \lim_{h\to 0} \frac{\frac{0}{h^2}-0}{h} = 0 \Rightarrow f_x(0,0) \text{ exist.}$$

$$f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} \Rightarrow f_y(0,0) = \lim_{k\to 0} \frac{\frac{0}{k^4}-0}{k} = 0 \Rightarrow f_y(0,0) \text{ exists.}$$

f is not differentiable at $(0,0)$ since $f(x,y)$ does not have the limit at $(0,0)$; (that is, f has discontinuity at $(0,0)$):

On the line $y=0$:

$\lim f(x,y) = 0$

On the line $y^2=x$:

$$f(y^2,y) = \frac{y^4}{y^4+y^4} = \frac{1}{2} \Rightarrow \lim f(x,y) = \frac{1}{2}$$

2. Find:

a) an equation of the plane tangent to the graph of the function $f(x,y) = \dfrac{x}{x^2+y^2}$ at the point $(1,2)$.

$$f_1(x,y) = \frac{y^2-x^2}{(x^2+y^2)^2} \qquad f_2(x,y) = -\frac{2y}{(x^2+y^2)^2}$$

$$\nabla f(x,y) = \text{grad } f(x,y) = f_1(x,y)\,i + f_2(x,y)\,j = \frac{y^2-x^2}{(x^2+y^2)^2}\,i - \frac{2y}{(x^2+y^2)^2} \Rightarrow \nabla f(1,2) = \frac{3}{25}i - \frac{4}{25}$$

$$z = f_1(x,y).(x-x_0) + f_2(x,y).(y-y_0) + f(x_0,y_0)$$

$$z = \frac{3}{25}.(x-1) - \frac{4}{25}.(y-2) + \frac{1}{5} \Rightarrow z = \frac{1}{25}(3x-4y+10)$$

$$\Rightarrow 3x-4y-25z = -10$$

b) an equation of the straight line tangent, at the point $(1,2)$, to the level curve of the function $f(x,y) = \dfrac{x}{x^2+y^2}$ passing through that point.

$$\frac{3}{25}(x-1) - \frac{4}{25}(y-2) = 0 \Rightarrow 3(x-1) - 4(y-2) = 0$$

$$\Rightarrow 3x - 4y + 5 = 0$$

c) the directional derivative of the function $f(x,y) = \dfrac{x}{x^2+y^2}$ at the point $(1,2)$ in the direction of the vector $2i + j$.

$$D_u f(1,2) = u \bullet \nabla f(1,2)$$

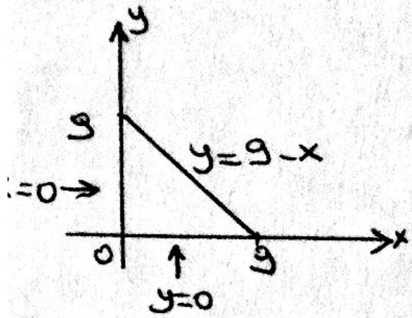
$$= \frac{2i+j}{\sqrt{2^2+1^2}} \bullet \left(\frac{3}{25}i - \frac{4}{25}j\right)$$

$$= \left(\frac{2}{\sqrt5}i + \frac{1}{\sqrt5}j\right) \bullet \left(\frac{3}{25}i - \frac{4}{25}j\right)$$

$$= \frac{2}{\sqrt5}\cdot\frac{3}{25} + \frac{1}{\sqrt5}\cdot\left(\frac{-4}{25}\right)$$

$$= \frac{2}{25\sqrt5}$$

3. Find the absolute maximum and minimum values of $f(x,y) = 2+2x+2y-x^2-y^2$ on the triangular region bounded by $x = 0, y = 0$ and $y = 9-x$.



$$\left.\begin{array}{l} f_1(x,y) = 2-2x = 0 \Rightarrow x = 1 \\ f_2(x,y) = 2-2y = 0 \Rightarrow y = 1 \end{array}\right\} \Rightarrow \text{The critical point is } (1,1).$$

$f(1,1) = 4$

<u>$x = 0;\ 0 \le y \le 9$</u>:

$f(0,y) = 2+2y-y^2$

Let $g(y) = 2+2y-y^2 \Rightarrow g'(y) = 2-2y = 0 \Rightarrow y = 1$

$g(0) = 2,\ g(9) = -61,\ g(1) = 3$

<u>$y = 0,\ 0 \le x \le 9$</u>:

$f(x,0) = 2+2x-x^2$

Let $h(x) = 2+2x-x^2$

$h'(x) = 2-2x = 0 \Rightarrow x = 1$

$h(0) = 2,\ h(9) = -61,\ h(1) = 3$

<u>$y = 9-x;\ 0 \le x \le 9$</u>:

$f(x,y) = f(x,9-x) = -61+18x-x^2$

Let $k(x) = -61+18x-2x^2$

$k'(x) = 18-4x = 0 \Rightarrow x = \frac{9}{2}$

$k(0) = -61,\ k\left(\frac{9}{2}\right) = -\frac{41}{2},\ k(9) = -61$

Therefore, $f(1,1) = 4$ abs. max

-61 abs min

4. (a) Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$ by using Lagrange multipliers.

$$L(x,y,z,\lambda) = xyz + \lambda\cdot(x^2+y^2+z^2-12)$$

$$\frac{\partial L}{\partial x} = yz + 2\lambda x = 0 \quad (A) \qquad \frac{\partial L}{\partial z} = xy + 2\lambda z = 0 \quad (C)$$

$$\frac{\partial L}{\partial y} = xz + 2\lambda y = 0 \quad (B) \qquad \frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-12 = 0 \quad (D)$$

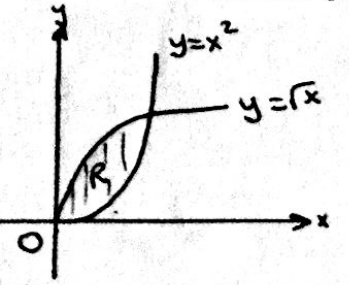
Multiplying equations (A), (B), (C) by x, y, z respectively and subtracting in pairs, we conclude that $\lambda x^2 = \lambda y^2 = \lambda z^2$ so that either $\lambda = 0$ or $x^2 = y^2 = z^2$.

If $\lambda = 0$, (A), then $yz = 0$ so $xyz = 0$.

If $x^2 = y^2 = z^2$ then equation (D) gives $3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.

At four of these points $xyz = 8$ which is the max. value of xyz on the sphere. At other four $xyz = -8$ which is the minimum value.

(b) Evaluate the double integral $\iint_R (x^2 + y^2)\,dA$, where R is the finite region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$.



$$\iint_R (x^2+y^2)\,dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)\,dy\,dx$$

$$= \int_0^1 \left(x^2 y + \frac{y^3}{3}\right)\Bigg|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[x^2\cdot(\sqrt{x} - x^2) + \frac{1}{3}\cdot\left((\sqrt{x})^3 - (x^2)^3\right)\right] dx$$

$$= \int_0^1 \left[x^{5/2} - x^4 + \frac{1}{3}\cdot x^{3/2} - \frac{x^6}{3}\right] dx$$

$$= \left(\frac{2}{7}x^{7/2} - \frac{x^5}{5} + \frac{2}{15}x^{5/2} - \frac{x^7}{21}\right)\Bigg|_0^1$$

$$= \frac{2}{7} - \frac{1}{5} + \frac{2}{15} - \frac{1}{21}$$

$$= \frac{6}{35}$$