25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II **03.06.2011**

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

Name: ....#KEY#........

Student No: ....

Department: ....

Section: **Check for your instructor below:**

- ☐ Tahsin Öner
- ☐ Aslı Güldürdek
- ☐ Sevin Gümgüm
- ☐ Ebru Özbilge

1. (a) Evaluate the given double integral over the quarter-disk Q given by $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 4$.

$$\iint_Q \frac{xy}{x^2+y^2} dA.$$

By using Polar coordinates:

$$\iint_Q \frac{xy}{x^2+y^2} dA = \int_0^{\pi/2} \int_0^2 \frac{r\cos\theta \cdot r\sin\theta}{r^2} r \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \int_0^2 r \sin 2\theta \, dr \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/2} \left(\frac{r^2}{2} \Big|_0^2 \right) \sin 2\theta \, d\theta = \int_0^{\pi/2} \sin 2\theta \, d\theta$$

$$= -\frac{1}{2} \cos 2\theta \Big|_0^{\pi/2} = -\frac{1}{2} [\cos\pi - \cos 0] = 1$$

(b) Evaluate the triple integral $\iiint_R xy^2 e^{-xyz} dV$ over the cube $0 \leq x, y, z \leq 1$.

$$\iiint_R xy^2 e^{-xyz} dV = \int_0^1 \int_0^1 \int_0^1 xy^2 e^{-xyz} dz \, dx \, dy$$

$$= \int_0^1 \int_0^1 xy^2 \frac{e^{-xyz}}{-xy} \Big|_0^1 dx \, dy = \int_0^1 \int_0^1 y(1 - e^{-xy}) dx \, dy$$

$$= \int_0^1 \int_0^1 (y - ye^{-xy}) dx \, dy = \int_0^1 \left(xy - y \frac{e^{-xy}}{-y} \right) \Big|_0^1 dy$$

$$= \int_0^1 (xy + e^{-xy}) \Big|_0^1 dy = \int_0^1 (y + e^{-y} - 1) dy$$

$$= \left(\frac{y^2}{2} - e^{-y} - y \right) \Big|_0^1 = \frac{e-1}{2e}$$

2. (a) Find the volume of the region inside the paraboloid $z = x^2 + y^2$ and inside the sphere $x^2 + y^2 + z^2 = 20$.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad z = x^2+y^2 \Rightarrow z = r^2 \qquad x^2+y^2+z^2 = 20 \Rightarrow r^2 + r^4 = 20$$

$$\Rightarrow r^4 + r^2 - 20 = 0 \Rightarrow (r^2+5)(r^2-4) = 0$$

$$\Rightarrow r = 2$$

$$V = \iiint dV = \int_0^{2\pi}\int_0^{2}\int_{r^2}^{\sqrt{20-r^2}} r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{2} r.(\sqrt{20-r^2} - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^{2} (r\sqrt{20-r^2} - r^3)\, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(20-r^2)^{3/2} - \frac{r^4}{4}\right]\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} \left[\left(-\frac{1}{3}.64 - 4\right) - \left(-\frac{1}{3}.20^{3/2}\right)\right] d\theta = \int_0^{2\pi}\left[-\frac{76}{3} + \frac{1}{3}.20^{3/2}\right] d\theta$$

$$= 2\pi.\left(-\frac{76}{3} + \frac{1}{3}.20^{3/2}\right)$$

(b) Find $\iiint_B x^2+y^2+z^2 dV$ where B is the upper-half of the ball $x^2+y^2+z^2 = 1$.

Spherical coordinates

$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$$

$$x^2+y^2+z^2 = \rho^2$$

$$\iiint_B (x^2+y^2+z^2)\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1} \rho^2.\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{5}\sin\phi\, d\phi\, d\theta = \frac{1}{5}\int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi/2} d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\left[-\cos\frac{\pi}{2} + \cos 0\right] d\theta = \frac{1}{5}\int_0^{2\pi} d\theta$$

$$= \frac{2\pi}{5}$$

3. (a) Verify that $y = cos3x$ and $y = sin3x$ are the solutions of the differential equation $y'' + 9y = 0$. Are any of the following functions solutions?
(i) $y = 3sin3x - 9cos3x$, and (ii) $y = sin6x$.

$y = \cos(3x) \Rightarrow y' = -3\sin(3x)$ and $y'' = -9\cos(3x)$

$y'' + 9y = -9\cos(3x) + 9\cdot\cos(3x) = 0$ Thus, $y = \cos(3x)$ is a sln.

$y = \sin(3x) \Rightarrow y' = 3\cos(3x)$ and $y'' = -9\sin(3x)$

$y'' + 9y = -9\sin(3x) + 9\cdot\sin(3x) = 0$ thus, $y = \sin(3x)$ is a sln.

$y = 3\sin(3x) - 9\cos(3x) \Rightarrow y' = 9\cos(3x) + 27\sin(3x)$

$y'' = -27\sin(3x) + 81\cos(3x)$

$y'' + 9y = -27\sin(3x) + 81\cos(3x) + 9(3\sin(3x) - 9\cos(3x)) = 0$

$y = \sin(6x) \Rightarrow y' = 6\cos(6x)$ and $y'' = -36\sin(6x)$

$y'' + 9y = -36\sin(6x) + 9\sin(6x) \neq 0$ Thus $y = \sin(6x)$ is not a solution.

(b) Solve the initial-value problem

$$\begin{cases} y' + (sinx)y = 2xe^{cosx} \\ y(\frac{\pi}{2}) = 1. \end{cases}$$

$p(x) = \sin x$, $q(x) = 2xe^{\cos x}$

$\mu(x) = \int p(x)dx = \int \sin x dx = -\cos x$ and $e^{\mu(x)} = e^{-\cos x}$

Since the DE is a first-order linear nonhomogeneous, the solution is

$y = e^{-\mu(x)} \int e^{\mu(x)} \cdot q(x) dx$

$y = e^{\cos x} \int e^{-\cos x} \cdot 2xe^{\cos x} dx$

$y = e^{\cos x} \int 2x dx = e^{\cos x}(x + c)$

$y(\frac{\pi}{2}) = 1 \Rightarrow 1 = \underbrace{e^{\cos\frac{\pi}{2}}}_{=1}(\frac{\pi}{2} + c) \Rightarrow c = 1 - \frac{\pi}{2}$

$\therefore y = e^{\cos x}(x + 1 - \frac{\pi}{2})$

4. (a) Solve the differential equation $x\frac{dy}{dx} = x\tan(\frac{y}{x}) + y$.

$x \cdot \frac{dy}{dx} = x \cdot \tan\frac{y}{x} + y \Rightarrow \frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x}$ homogeneous eqn.

$v = \frac{y}{x} \Rightarrow y = v \cdot x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} \cdot x + v$

$\frac{dy}{dx} = \tan\frac{y}{x} + \frac{y}{x} \Rightarrow \frac{dv}{dx} \cdot x + v = \tan v + v \Rightarrow \frac{dv}{dx} \cdot x = \tan v$

$\Rightarrow \cot v \cdot dv = \frac{dx}{x} \Rightarrow \ln|\sin v| = \ln|x| + \ln C \Rightarrow \ln|\sin v| = \ln|Cx|$

$\Rightarrow \sin v = Cx \Rightarrow \sin\left(\frac{y}{x}\right) = Cx$ or $y = x \cdot \sin^{-1}(Cx)$

(b) Solve the differential equation $(\frac{1}{x^2} + \frac{1}{y^2})dx + (\frac{-2x+1}{y^3})dy = 0$.

$M(x,y)dx + N(x,y)dy = 0$ $\quad \frac{\partial M}{\partial y} = -\frac{2}{y^3} = \frac{\partial N}{\partial x}$ exact.

we want to find $\phi(x,y) = C$ such that $\frac{\partial \phi}{\partial x} = M$ and $\frac{\partial \phi}{\partial y} = N$

$\frac{\partial \phi}{\partial x} = M \Rightarrow \phi(x,y) = \int M(x,y)dx + C_1(y)$

$= \int \left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + C_1(y) = \frac{1}{x^2} + \frac{x}{y^2} + C_1(y)$

$\frac{\partial \phi}{\partial y} = N \Rightarrow -\frac{2\cancel{x}}{\cancel{y^3}} + C_1'(y) = -\frac{2\cancel{x}}{\cancel{y^3}} + \frac{1}{y^3} \Rightarrow C_1(y) = \int \frac{1}{y^3}dy$

$\Rightarrow C_1(y) = -\frac{1}{2y^2} + C_2$

$\phi(x,y) = -\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_2 = C$

or

$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = k$