25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 09.06.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

Name: ....

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Duration: 110 mins

1. (a) Solve the differential equation $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + 2y^2}$.

Solution

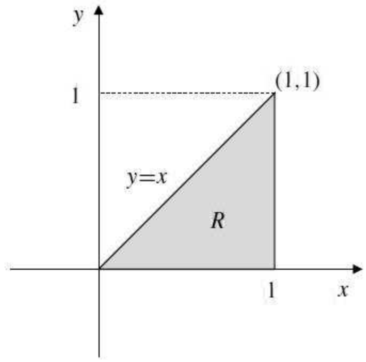
$$
\begin{aligned}
\frac{dy}{dx} &= \frac{xy}{x^2 + 2y^2} \quad \text{Let } y = vx \\
v + x\frac{dv}{dx} &= \frac{vx^2}{(1 + 2v^2)x^2} \\
x\frac{dv}{dx} &= \frac{v}{1 + 2v^2} - v = -\frac{2v^3}{1 + 2v^2} \\
\int \frac{1 + 2v^2}{v^3}\, dv &= -2\int \frac{dx}{x} \\
-\frac{1}{2v^2} + 2\ln|v| &= -2\ln|x| + C_1 \\
-\frac{x^2}{2y^2} + 2\ln|y| &= C_1 \\
x^2 - 4y^2 \ln|y| &= Cy^2.
\end{aligned}
$$

(b) Solve the differential equation $(e^x \cos y + 2x)dx + (e^{-x} \sin y + 2y)dy = 0$.

2. (a) Evaluate the iterated integral $\int_0^1 dy \int_y^1 e^{-x^2} dx$.

Solution

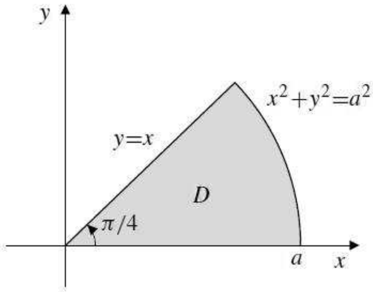
$$\begin{aligned}
\int_0^1 dy \int_y^1 e^{-x^2} dx &= \int_R e^{-x^2} dA \quad (R \text{ as shown}) \\
&= \int_0^1 e^{-x^2} dx \int_0^x dy \\
&= \int_0^1 x e^{-x^2} dx \quad \text{Let } u = x^2, \quad du = 2x dx \\
&= \frac{1}{2} \int_0^1 e^{-u} du = -\frac{1}{2} e^{-u} \Big|_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right).
\end{aligned}$$



(b) Evaluate $\iint_D xy\, dA$, where D is the plane region satisfying $x \geq 0$, $0 \leq y \leq x$, and $x^2 + y^2 \leq a^2$.

Solution

$$\begin{aligned}
\iint_D xy dA &= \int_0^{\pi/4} d\theta \int_0^a r\cos\theta r \sin\theta r dr \\
&= \frac{1}{2} \int_0^{\pi/4} \sin 2\theta d\theta \int_0^a r^3 dr \\
&= \frac{a^4}{8}\left(-\frac{\cos 2\theta}{2}\right)\Big|_0^{\pi/4} = \frac{a^4}{16}.
\end{aligned}$$



3. (a) Find the area of the region in the first quadrant bounded by the curves $xy = 1$, $xy = 4$, $y = x$ and $y = 2x$.

Solution

Let $u = xy$, $v = y/x$. Then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -y/x^2 & 1/x \end{vmatrix} = 2\frac{y}{x} = 2v,$$

so that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2v}$. The region D in the first quadrant of the xy-plane bounded by $xy = 1$, $xy = 4$, $y = x$, and $y = 2x$ corresponds to the rectangle R in the uv-plane bounded $u = 1$, $u = 4$, $v = 1$, and $v = 2$. Thus the area of D is given by

$$\iint_D dxdy = \iint_R \frac{1}{2v} dudv = \frac{1}{2}\int_1^4 du \int_1^2 \frac{dv}{v} = \frac{3}{2}\ln 2 \text{ sq. units.}$$

(b) Evaluate the triple integral $\iiint_R y\,dV$, where R is the tetrahedron bounded by the coordinate planes and the plane $x + y + z = 1$.

Solution

$$\int_0^1 \int_0^{1-x} \int_0^{1-(x+y)} y\,dzdydx = \int_0^1 \int_0^{1-x} y(z\big|_0^{1-(x+y)})dydx$$

$$= \int_0^1 \int_0^{1-x} (y - xy - y^2)dydx = \int_0^1 (\frac{y^2}{2} - \frac{xy^2}{2} - \frac{y^3}{3})\big|_0^{1-x} dx$$

$$= \int_0^1 \frac{(1-x)^3}{6} dx = \frac{-(1-x)^4}{24}\big|_0^1 = \frac{1}{24}$$

4. (a) Use cylindrical coordinates to evaluate the volume of the region between paraboloids $z = 16 - x^2 - y^2$ and $z = x^2 + y^2 - 2$.

Solution

$$x = r\sin\theta, \quad y = r\cos\theta, \quad z = z$$

$$16 - x^2 - y^2 = x^2 + y^2 - 2, \quad x^2 + y^2 = 9, \quad r = 3$$

$$\iint_R (16 - x^2 - y^2 - (x^2 + y^2 - 2))dA = \int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta$$

$$= \int_0^{2\pi} (9r^2 - \frac{r^4}{2})\Big|_0^3 d\theta = \int_0^{2\pi} \frac{81}{2} d\theta = 81\pi \text{ cubic units.}$$

(b) Find $\iiint_B (x^2 + y^2)\, dV$, where B is the ball given by $x^2 + y^2 + z^2 \leq a^2$.

Solution

$$\iiint_B (x^2 + y^2)\, dV$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi} \sin\phi\, d\phi \int_0^a R^2 \sin^2\phi R^2\, dR$$

$$= 2\pi \int_0^{\pi} \sin^3\phi\, d\phi \int_0^a R^4\, dR = 2\pi \frac{a^5}{5} \int_0^{\pi} \sin^3\phi\, d\phi = 2\pi \frac{a^5}{5} \int_0^{\pi} (1 - \cos^2\phi) \sin\phi\, d\phi$$

$$(\cos\theta = u, \quad \sin\theta d\theta = -du)$$

$$= 2\pi \frac{a^5}{5} \int_1^{-1} (u^2 - 1)\, du = 2\pi \frac{a^5}{5} (\frac{u^3}{3} - u)\Big|_1^{-1}$$

$$= 2\pi \left(\frac{4}{3}\right) \frac{a^5}{5} = \frac{8\pi a^5}{15}.$$