25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 02.05.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

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1. (a) Find the domain of $f(x, y) = \dfrac{1}{\sqrt{9 - x^2 - y^2}}$

Solution :

$9 - x^2 - y^2 > 0 \Rightarrow x^2 + y^2 < 9$

So, the domain of the function f is the disk $x^2 + y^2 < 9$ in the xy-plane.

(b) Is the function

$$f(x, y) = \begin{cases} \sin(xy) & ; \quad (x, y) \neq (0, 0) \\ 1 & ; \quad (x, y) = (0, 0) \end{cases}$$

continuous at $(0, 0)$? Why?

Solution :

$\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,y)\to(0,0)} \sin(xy) = \sin 0 = 0$

But $f(0, 0) = 1$.

So, $\lim\limits_{(x,y)\to(0,0)} f(x, y) \neq f(0, 0)$. Therefore, f is not continuous at the given point $(0, 0)$.

2. (a) Find the partial derivatives of

$$f(x,y) = \begin{cases} \dfrac{x^5 - y^5}{x^4 + y^4} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$$

with respect to x and y at $(0,0)$.

Solution :

$$\begin{aligned} f_1(0,0) &= \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \to 0} \frac{\frac{h^5-0}{h^4+0} - 0}{h} \\ &= \lim_{h \to 0} \frac{h^5}{h^5} \\ &= 1\ , \end{aligned}$$

$$\begin{aligned} f_2(0,0) &= \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} \\ &= \lim_{h \to 0} \frac{f(0,k) - f(0,0)}{k} \\ &= \lim_{h \to 0} \frac{\frac{0-k^5}{0+k^4} - 0}{k} \\ &= \lim_{h \to 0} \frac{-k^5}{k^5} \\ &= -1\ , \end{aligned}$$

(b) Find the rate of change of $f(x,y) = x^2y^2$ at $(-2,-2)$ in the direction of the vector $i + 2j$.

Solution :

$$\begin{aligned} \nabla f(x,y) &= 2xy^2 i + 2x^2 y j \\ \nabla f(-2,-2) &= -16i - 16j \end{aligned}$$

$$\begin{aligned} D_u f(-2,-2) &= \frac{(i+2j)}{\sqrt{5}} \cdot (-16i - 16j) \\ &= \frac{-16 + 2(-16)}{\sqrt{5}} \\ &= \frac{-48}{\sqrt{5}}. \end{aligned}$$

3. (a) Find the maximum value of $f(x,y) = xy - x^3y^2$ over the square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

Solution :

$f(x,y) = xy - x^3y^2$ on the square S: $0 \leq x \leq 1$, $0 \leq y \leq 1$.

$$\begin{aligned} f_1 &= y - 3x^2y^2 = y(1-3x^2y) = 0 \\ f_2 &= x - 2x^3y = x(1-2x^2y) = 0 \end{aligned}$$

$(0,0)$ is a critical point. Any other critical points must satisfy $3x^2y = 1$ and $2x^2y = 1$ that is $x^2y = 0$. Therefore $(0,0)$ is the only critical point, and it is on the boundary of S. We need therefore only consider the values of f on the boundary of S.

On the sides $x = 0$ and $y = 0$ of S, $f(x,y) = 0$. On the side $x = 1$, we have $f(1,y) = y - y^2 = g(y)$, $0 \leq y \leq 1$. The function g has maximum $\frac{1}{4}$ at its critical point $y = \frac{1}{2}$.

On the side $y = 1$ we have $f(x,1) = x - x^3 = h(x)$, $0 \leq x \leq 1$, h has critical point given by $1 - 3x^2 = 0$; only $x = \frac{1}{\sqrt{3}} = \frac{2}{3\sqrt{3}} > \frac{1}{4}$.

On the square S, $f(x,y)$ has minimum value 0 (on the sides $x = 0$, $y = 0$ and at the corner $(1,1)$ of the square) and maximum value $\frac{2}{3\sqrt{3}}$ at the point $(\frac{1}{\sqrt{3}}, 1)$. There is a smaller local maximum value at $(1, \frac{1}{2})$.

(b) Find and classify the critical points of $f(x,y) = \dfrac{y}{x} + \dfrac{8}{y} - x$.

Solution :

$f(x,y) = \dfrac{y}{x} + \dfrac{8}{y} - x$,

$$f_1(x,y) = -\frac{y}{x^2} - 1 = 0 \qquad (1)$$

$$f_2(x,y) = \frac{1}{x} - \frac{8}{y^2} = 0 \qquad (2)$$

From (1), we have $-y = x^2$ and From (2), we have $y^2 = 8x$

$x^4 = 8x \quad \Rightarrow x^3 = 8 \quad \Rightarrow x = 2$ and $y = -4$ Also, $x = 0 \quad \Rightarrow y = 0$.

For, $x = 0$ and $y = 0$, $f(x,y)$ is undefined. So only critical point is $(2,-4)$.

$$\begin{aligned} f_{11}(x,y) &= \frac{2y}{x^3} \\ f_{22}(x,y) &= \frac{16}{y^3} \\ f_{12}(x,y) &= f_{21}(x,y) = -\frac{1}{x^2} \end{aligned}$$

Hence,

$$A = f_{11}(2,-4) = -1, \quad B = f_{12}(2,-4) = -\frac{1}{4}, \quad C = f_{22}(2,-4) = -\frac{1}{4}$$

then

$$B^2 - AC = \left(\frac{-1}{4}\right)^2 - \left(\frac{-1}{4}\right).(-1) = \left(\frac{1}{16}\right) - \left(\frac{1}{4}\right) < 0$$

and since $A < 0$, $(2,-4)$ is a local maximum.

4. (a) Use the method of Lagrange multipliers to find the shortest distance from the origin to the surface $xyz^2 = 2^4$

Solution :

$$L = x^2 + y^2 + z^2 + \lambda(xyz^2 - 2^4)$$

For, critical points, we need to solve

$$0 = \frac{\partial L}{\partial x} = 2x + \lambda yz^2 \Rightarrow -\lambda xyz^2 = 2x^2 \quad (3)$$

$$0 = \frac{\partial L}{\partial y} = 2y + \lambda xz^2 \Rightarrow -\lambda xyz^2 = 2y^2 \quad (4)$$

$$0 = \frac{\partial L}{\partial z} = 2z + 2\lambda xyz \Rightarrow -\lambda xyz^2 = z^2 \quad (5)$$

$$0 = \frac{\partial L}{\partial \lambda} = xyz^2 - 2^4, \quad (6)$$

By (3) – (5), we find $x^2 = y^2$ and $z^2 = 2x^2$. From (6), we have $x^2y^2z^4 = 2^8$ or $4x^8 = 2^8$. Thus $x^2 = y^2 = 2^{\frac{3}{2}}$ and $z^2 = 2^{\frac{5}{2}}$. The shortest distance from the origin to the surface $xyz^2 = 2^4$ is

$$\sqrt{2^{\frac{3}{2}} + 2^{\frac{3}{2}} + 2^{\frac{5}{2}}} = 2^{\frac{7}{4}} \quad \text{units}$$

(b) Evaluate $\int_R\int \frac{\sin y}{y} dA$, where $R = \{(x, y) : x \le y \le \pi/2, \ 0 \le x \le \pi/2\}$

Solution :

$$\begin{aligned}\int_R\int \frac{\sin y}{y} dA &= \int_0^{\frac{\pi}{2}} dx \int_x^{\frac{\pi}{2}} \frac{\sin y}{y} dy \\ &= \int_0^{\frac{\pi}{2}} dy \int_0^{y} \frac{\sin y}{y} dx \\ &= \int_0^{\frac{\pi}{2}} \sin y dy \\ &= -\cos y\Big|_0^{\frac{\pi}{2}} = 1\end{aligned}$$