25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 154 CALCULUS II 27.03.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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Duration: 110 mins

1. (a) Evaluate the improper integral or show that it diverges

$$\int_1^{\infty} \frac{dx}{x\sqrt[3]{\ln x}}$$

Solution :

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x\sqrt[3]{\ln x}} &= \int_1^{c} \frac{dx}{x(\ln x)^{1/3}} + \int_c^{\infty} \frac{dx}{x(\ln x)^{1/3}} \\ &= \lim_{a\to 1^-} \int_a^{c} \frac{dx}{x(\ln x)^{1/3}} + \lim_{b\to\infty} \int_c^{b} \frac{dx}{x(\ln x)^{1/3}}\end{aligned}$$

where $c \in (1, \infty)$ is any positive number. Let $\ln x = t,\ \ \frac{1}{x}\,dx = dt$ then

$$\lim_{a\to 1^-} \int_a^{c} \frac{dx}{x(\ln x)^{1/3}} + \lim_{b\to\infty} \int_c^{b} \frac{dx}{x(\ln x)^{1/3}} = \lim_{a\to 1^-} \int_{\ln a}^{\ln c} \frac{dt}{t^{1/3}} + \lim_{b\to\infty} \int_{\ln c}^{\ln b} \frac{dt}{t^{1/3}}$$

$$= \lim_{a\to 1^-} \left(\frac{3}{2}t^{2/3}\right)\Big|_{\ln a}^{\ln c} + \lim_{b\to\infty} \left(\frac{3}{2}t^{2/3}\right)\Big|_{\ln c}^{\ln b}$$

$$= \lim_{a\to 1^-} \frac{3}{2}[(\ln c)^{2/3} - (\ln a)^{2/3}] + \lim_{b\to\infty} \frac{3}{2}[(\ln b)^{2/3} - (\ln c)^{2/3}] = \infty$$

So, the integral diverges.

(b) Find the length of the curve $y = \frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 4$.

Solution : The arc length element is

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

where $\frac{dy}{dx} = y' = \frac{x^2}{4} - \frac{1}{x^2}$ then,

$$ds = \sqrt{1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2}\,dx = \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx$$

and the length

$$L = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right) dx = \left(\frac{x^3}{12} - \frac{1}{x}\right)\Big|_1^4 = 6 \ \text{ units.}$$

2. Find the volume of the solid obtained by rotating the plane region R bounded by $y = x(3-x)$ and $y = 0$ between $x = 0$ and $x = 3$ about

(a) the x-axis using plane slices

(b) the y-axis using cylindrical shells

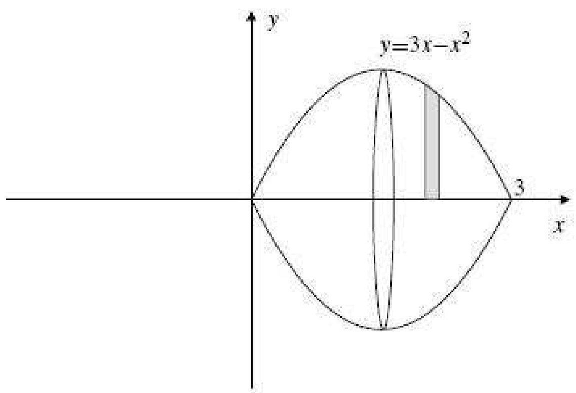
Solution :

(a) If the region R bounded by $y = x(3-x)$ is rotated about the x-axis, to find the volume of the obtained solid we use the method of slicing. Here, $f(x) = x(3-x)$. So,

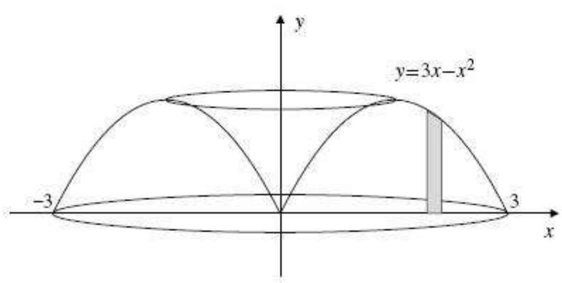
$$\begin{aligned} V &= \pi \int_0^3 [x(3-x)]^2 dx \\ &= \pi \int_0^3 (9x^2 - 6x^3 + x^4) dx \\ &= \pi (3x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5)\Big|_0^3 = \frac{81\pi}{10} \text{ cu.units} \end{aligned}$$

(b) If the region R bounded by $y = x(3-x)$ is rotated about the y-axis, to find the volume of the obtained solid we use the method of cylindrical shells. So,

$$\begin{aligned} V &= 2\pi \int_0^3 x^2(3-x) dx \\ &= 2\pi \int_0^3 (3x^2 - x^3) dx \\ &= 2\pi (x^3 - \frac{1}{4}x^4)\Big|_0^3 = \frac{27\pi}{2} \text{ cu.units} \end{aligned}$$



(a)



(b)

3. Test the given series for convergence

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$

Solution : We apply the ratio test

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \left[\frac{(n+1)^{n+1}}{3^{n+1}(n+1)!} \cdot \frac{3^n n!}{n^n}\right] = \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \frac{e}{3} < 1$$

Hence, according to the ratio test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{3^n n!}$ converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[(n+\frac{1}{2})\pi\right]}{2n+3}$

Solution : $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[\left(n+\frac{1}{2}\right)\pi\right]}{2n+3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n+3}$

We apply the alternating series test, since

- $a_{n+1} \cdot a_n < 0$;
- $|a_{n+1}| < |a_n|$;
- $\displaystyle\lim_{n\to\infty} \frac{(-1)^n}{2n+3} = 0$

So, it converges. But the series does not converges absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{\sin\left[\left(n+\frac{1}{2}\right)\pi\right]}{2n+3} \right| = \sum_{n=1}^{\infty} \frac{1}{2n+3}$$

and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$ diverges. Hence, the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left[\left(n+\frac{1}{2}\right)\pi\right]}{2n+3}$ converges conditionally.

4. (a) Find the Maclourin series of $\dfrac{x(1+x)}{(1-x)^3}$ by using the representation $\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$.

Solution :

By differentiating both sides of $\dfrac{x}{(1-x)^2} = \sum_{n=1}^{\infty} nx^n$, we have

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}$$

Multiplying both sides with x yields

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}.$$

(b) Find the Taylor series representation of $\dfrac{x+1}{3x-1}$ in powers of $x+1$. Where does the series converge?

Solution : Let $x+1=t$ and $x=t-1$, then

$$\frac{x+1}{3x-1} = \frac{t}{3(t-1)-1} = \frac{t}{3t-4} = \frac{-t}{4-3t} = \frac{-t}{4}\left(\frac{1}{1-\frac{3t}{4}}\right)$$

Since, $\dfrac{1}{1-u} = \left(1+u+u^2+u^3+\ldots\right)$, then

$$\begin{aligned}\frac{-t}{4}\left(\frac{1}{1-\frac{3t}{4}}\right) &= \frac{-t}{4}\left(1+(\frac{3t}{4})+(\frac{3t}{4})^2+(\frac{3t}{4})^3+\ldots\right)\\ &= \frac{-t}{4}\sum_{n=0}^{\infty}\left(\frac{3t}{4}\right)^n = -\sum_{n=0}^{\infty}\frac{3^n t^{n+1}}{4^{n+1}}\end{aligned}$$

Hence, $\dfrac{x+1}{3x-1} = \sum_{n=0}^{\infty} \dfrac{3^n (x+1)^{n+1}}{4^{n+1}}$

Interval of the convergence: $\left|\dfrac{3t}{4}\right| < 1$ then

$$-1 < \frac{3t}{4} < 1 \quad \Rightarrow \quad -\frac{4}{3} < t < \frac{4}{3} \quad \Rightarrow \quad -\frac{4}{3} < x+1 < \frac{4}{3} \quad \Rightarrow \quad -\frac{7}{3} < x < \frac{1}{3}.$$