1. (a) Starting with the power series representation

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n$$

determine the Taylor series representation of $f(x) = \dfrac{1}{x^2}$ in powers of $x - 5$.

Solution:

First, differentiate the function $\dfrac{1}{1-x}$ and its series representation with respect to x to get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + ... = \sum_{n=1}^{\infty} n\, x^{n-1}.$$

Let $t = x - 5$, i.e., $x = 5 + t$. So, we have

$$f(x) = \frac{1}{x^2} = \frac{1}{(5+t)^2} = \frac{1}{25\left(1 + \frac{t}{5}\right)^2}.$$

Substitute $\dfrac{-t}{5}$ for x in $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n\, x^{n-1}$, to get

$$\frac{1}{\left(1 + \frac{t}{5}\right)^2} = \sum_{n=1}^{\infty} n \left(\frac{-t}{5}\right)^{n-1}.$$

Then multiply the resulting equation by $\dfrac{1}{25}$ to arrive at

$$\frac{1}{25\left(1 + \frac{t}{5}\right)^2} = \frac{1}{25} \sum_{n=1}^{\infty} n \left(\frac{-t}{5}\right)^{n-1}.$$

Finally, use the transformation $t = x - 5$ to find

$$\frac{1}{x^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{5^{n+1}} (x-5)^{n-1}.$$

(b) If $S(x) = \int_0^x \sin(t^2)\, dt$, find $\lim_{x \to 0} \dfrac{x^3 - 3S(x)}{x^7}$

Solution:

Maclaurin series for $\sin(t^2) = t^2 - \dfrac{t^6}{3!} + \dfrac{t^{10}}{5!} - ...$

$$S(x) = \int_0^x \sin(t^2)\, dt = \int_0^x \left(t^2 - \frac{t^6}{3!} + \frac{t^{10}}{5!} - ...\right) dt$$

$$S(x) = \left(\frac{t^3}{3} - \frac{t^7}{7 \times 3!} + \frac{t^{11}}{11 \times 5!} - ...\right)\Bigg|_0^x$$

$$S(x) = \frac{x^3}{3} - \frac{x^7}{7 \times 3!} + \frac{x^{11}}{11 \times 5!} - ...$$

Substitute $S(x)$ in the limit:

$$\lim_{x\to 0} \frac{x^3 - 3S(x)}{x^7} = \lim_{x\to 0} \frac{x^3 - 3\Big(\frac{x^3}{3} - \frac{x^7}{7\times 3!} + \frac{x^{11}}{11\times 5!} - ...\Big)}{x^7}$$

$$= \lim_{x\to 0} \Big(\frac{3}{7\times 3!} - \frac{3\,x^4}{11\times 5!} + ...\Big) = \frac{3}{7\times 3!} = \frac{1}{14}$$

2. Find the Fourier series of the function $f(t)$ with period 2 whose values in the interval $[-1, 1)$ are given by

$$f(t) = \begin{cases} 0 & \text{if } 1 \le t < 0 \\ t & \text{if } 0 \le t < 1 \end{cases}.$$

Solution:
The Fourier coefficients of f are as follows:

$$\frac{a_0}{2} = \frac{1}{2}\int_{-1}^{1} f(t)dt = \frac{1}{2}\int_{0}^{1} t dt = \frac{1}{4},$$

$$\begin{aligned} a_n &= \int_{-1}^{1} f(t)\cos(n\pi t)dt \\ &= \int_{0}^{1} t\cos(n\pi t)dt \\ &= \frac{(-1)^n - 1}{n^2\pi^2} \\ &= \begin{cases} -2/(n\pi)^2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}, \end{aligned}$$

and

$$b_n = \int_{0}^{1} t\sin(n\pi t)dt = \frac{-(-1)^n}{n\pi}.$$

Hence, the Fourier series of f is

$$\frac{1}{4} - \frac{2}{\pi^2}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}\cos((2k-1)\pi t) - \frac{1}{\pi}\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\sin(k\pi t).$$

3. (a) Find the general solution to $x\dfrac{dy}{dx} + 3y = 6x^3$.

Solution:

$$\frac{dy}{dx} + \frac{3}{x}y = 6x^2.$$

$$\mu(x) = e^{\int \frac{3}{x}dx} = x^3.$$

Multiply both sides by $\mu(x)$ to obtain

$$x^3\frac{dy}{dx} + 3x^2 y = 6x^5.$$

That is, $(x^3 y)' = 6x^5$. Integrate both sides to get $y = x^3 + Cx^{-3}$.

(b) Solve the integral equation $y(x) = 2 + \int_0^x e^{-y(t)}dt$, where $y(0) = 2$.

Solution:
$\frac{dy}{dx} = e^{-y}$

$e^y dy = dx$

$e^y = x + C \Rightarrow y = \ln(x + C)$

$y(0) = \ln(C) = 2 \Rightarrow C = e^2$

$y = \ln(x + e^2)$

4. (a) Find the general solution of the following ordinary differential equation

$$e^x(y - x)dx + (1 + e^x)dy = 0.$$

Solution:
Let $M = e^x(y - x)$ and $N = (1 + e^x)$. Since

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = e^x,$$

the given equation is exact and there exists a solution ϕ s.t

$$\frac{\partial \phi}{\partial x} = M, \qquad \frac{\partial \phi}{\partial y} = N.$$

Then the solution is of the form (by integrating N with respect to y)

$$\phi(x, y) = y + ye^x + g(x), \tag{1}$$

where g is an arbitrary function of x. Differentiation of (1) with respect to x and equating resulting equation to N we obtain

$$g'(x) = -xe^x,$$

which implies that

$$g(x) = -xe^x + e^x + c.$$

Hence, the solution is obtained as

$$\phi(x, y) = y + ye^x - xe^x + e^x + c = 0.$$

(b) Find the general solution of the following homogeneous differential equation

$$x\frac{dy}{dx} = y - x\sin^2(\frac{y}{x}).$$

Solution:
The given differential equation is of the type homogeneous and use the substitution $v = \frac{y}{x}$ for

$$\frac{dy}{dx} = \frac{y}{x} - \sin^2(\frac{y}{x}),$$

which implies the equation as

$$v + x\frac{dv}{dx} = v - \sin^2(v).$$

Thus

$$-\frac{dv}{\sin^2(v)} = \frac{dx}{x}$$

which yields the solution as

$$\cot(\frac{y}{x}) = \ln(xc).$$