SOLUTIONS

1.

a) Rotate about the x-axis, use the slicing method

$V = \int_0^1 \pi[4 - \dfrac{1}{(1+x)^2}]dx = \pi\ (4x + \dfrac{1}{1+x})\Big|_0^1 = \frac{7\pi}{2}$

b) Rotate about the y-axis, use the cylindrical shell method

$V = 2\pi\ \int_0^1 x\left(2 - \dfrac{1}{1+x}\right)dx$
$= 2\pi\ \left(x^2\Big|_0^1 - \int_0^1 \dfrac{x}{1+x}dx\right)$
$= 2\pi\ \left[1 - \int_0^1 (1 - \dfrac{1}{1+x})dx\right]$
$= 2\pi\ (1 - (x - \ln(1+x))\Big|_0^1)$
$= 2\pi\ \ln 2$

2. a) For $x \neq y$, we have

$$f(x,y) = \frac{x^4 - y^4}{x^2 - y^2}$$

The latter expression has the value $2x^2$ at points of the line $x = y$. Therefore, we extend the definition of $f(x,y)$ so that $f(x,x) = 2x^2$ then the resulting function will be equal to $f(x,y) = x^2 + y^2$ everywhere, and continuous everywhere

b) $\lim_{h\to 0} \dfrac{f(h,0) - 0}{h} = \lim_{h\to 0} \dfrac{\sin h^3}{h^3} = \lim_{h\to 0} \dfrac{\cos h^3\ (3h^2)}{3h^2} = 1$

$\lim_{k\to 0} \dfrac{f(0,k) - 0}{k} = \lim_{k\to 0} \dfrac{\sin k^3}{k^3} = \lim_{k\to 0} \dfrac{\cos k^3\ (3k^2)}{3k^2} = 1$

3. $f(x,y) = \ln(x^3 + y^3)$

$$f_1(x,y) = \frac{3x^2}{x^3+y^3},\ f_1(1,2) = \frac{1}{3}$$
$$f_2(x,y) = \frac{3y^2}{x^3+y^3},\ f_2(1,2) = \frac{4}{3}$$

a) $\nabla f(1,2) = \frac{1}{3}i + \frac{4}{3}j$

b) $f(1,2) = \ln 9$, the point of tangency is $(1, 2, \ln 9)$. Equation of the tangent plane:

$z = \ln 9 + \frac{1}{3}(x-1) + \frac{4}{3}(y-2)$

c) $\frac{1}{3}(x-1) + \frac{4}{3}(y-2) = 0 \quad \Rightarrow x + 4y = 9$

d) Equation of the normal line:

$$\frac{x-1}{1/3} = \frac{y-2}{4/3} = \frac{z-\ln 9}{-1}$$

4. a) The point (x,y,z) must be a critical function Lagrangian function

$$L = x^2 + y^2 + z^2 + \lambda(x + 2y + 2z - 3).$$

To find these critical points we have

$$\frac{\partial L}{\partial x} = 2x + \lambda = 0$$
$$\frac{\partial L}{\partial y} = 2y + 2\lambda = 0$$
$$\frac{\partial L}{\partial z} = 2z + 2\lambda = 0$$
$$\frac{\partial L}{\partial \lambda} = x + 2y + 2z - 3 = 0.$$

The first three equations yields $y = z = -\lambda$, $x = -\lambda/2$. Substituting these into the fourth equation we get $\lambda = -2/3$, so that the critical point is $(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, whose distance from the origin is 1.

b) $f(x,y) = xye^{-x+y}$

$f_1(x,y) = y(1-x)e^{-x+y}$

$f_2(x,y) = x(1+y)e^{-x+y}$

$A = f_{11}(x,y) = (-2y + xy)e^{-x+y}$

$B = f_{12}(x,y) = (1 - x + y - xy)e^{-x+y}$

$C = f_{22}(x,y) = (2x + xy)e^{-x+y}$

Critical points are $(0,0)$ and $(1,-1)$.

At $(0,0)$: $A = 0$, $B = 1$ and $C = 0$, so it is a saddle point.

At $(1,-1)$: $A = e^{-2}$, $B = 0$ and $C = e^{-2}$, so it is a local minimum point.