MATH 154 CALCULUS II 09.05.2009

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SECOND MIDTERM EXAM SOLUTIONS

1. a) $z = \sqrt{3}\sqrt{x^2+y^2} \Rightarrow z = \sqrt{3}r \Rightarrow \dfrac{r}{z} = \dfrac{1}{\sqrt{3}} = \tan\phi \Rightarrow \phi = \dfrac{\pi}{6}$

$x^2+y^2+z^2 = 9 \Rightarrow \rho^2 = 9$ gives $\rho = 3$

$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^{3} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \;=\; \int_0^{\pi/6}\int_0^{2\pi} \left(\frac{\rho^3}{3}\Big|_0^3\right)\sin\phi\, d\theta\, d\phi \;=\; 9\int_0^{\pi/6}\left(\theta\Big|_0^{2\pi}\right)\sin\phi\, d\phi$$

$$= 18\pi\left(-\cos\phi\Big|_0^{\pi/6}\right) \;=\; 18\pi(1-\sqrt{3}/2)$$

b) $[\rho,\phi,\theta] = [2, \pi/6, 2\pi/6]$

$$x = \rho\cos\theta\sin\phi = 2\,\frac{1}{2}\,\frac{1}{2} = \frac{1}{2}$$

$$y = \rho\sin\theta\sin\phi = 2\,\frac{\sqrt{3}}{2}\,\frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$z = \rho\cos\phi = 2\,\frac{\sqrt{3}}{2} = \sqrt{3}$$

Cartesian coordinates : $(x, y, z) = (1/2, \sqrt{3}/2, \sqrt{3})$

Cylindrical coordinates : $[r, \theta, z] = [1, 2\pi/6, \sqrt{3}]$

2. $x = 0$ is the center of convergence.

$$\lim_{n\to\infty} \left| \frac{\dfrac{n+1}{4^{n+1}\left((n+1)^2+1\right)}}{\dfrac{n}{4^n\left(n^2+1\right)}} \right| = \lim_{n\to\infty} \left| \frac{n+1}{4n} \frac{n^2+1}{n^2+2n+2} \right| = \frac{1}{4}$$

Radius of convergence : $R = 4$.

The series converges absolutely on $(-4, 4)$.

At $x = -4$: $\sum_{n=0}^{\infty} \dfrac{(-1)^n\, n}{n^2+1}$ converges conditionally (for absolute convergence use limit comparison test, compare with $\sum_{n=0}^{\infty} \dfrac{1}{n}$, for convergence use alternating series test)

At $x = 4$: $\sum_{n=0}^{\infty} \dfrac{n}{n^2+1}$ diverges (use limit comparison test, compare with $\sum_{n=0}^{\infty} \dfrac{1}{n}$)

3. a) Converges by integral test :

$$\int_3^{\infty} \frac{1}{t\,\ln t\,(\ln(\ln t))^3}\,dt = \lim_{R\to\infty} \int_3^{R} \frac{1}{t\,\ln t\,(\ln(\ln t))^3}\,dt$$

Let $u = \ln(\ln t)$ then $du = \dfrac{1}{t\,\ln t} dt$

$$\lim_{R\to\infty} \int \frac{1}{u^3}\,du = \lim_{R\to\infty} \left(\frac{-1}{2(\ln(\ln t))^2}\right)\Bigg|_3^R = \frac{1}{2(\ln(\ln 3))^2}$$

b) Converges by ratio test:

$$\lim_{n\to\infty} \left| \frac{\dfrac{(n+1)^{n+1}}{\pi^{n+1}\,(n+1)!}}{\dfrac{n^n}{\pi^n\, n!}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^n\,(n+1)}{\pi^n \pi\,(n+1)n!} \frac{n!\,\pi^n}{n^n} \right| = \frac{1}{\pi} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = \frac{e}{\pi}$$

4. a) Increasing :$n = 1$: $a_2 = \sqrt{21} > 3 = a_1$

$n = k$: $a_{k+1} > a_k$

$n = k+1$: $a_{k+2} > a_{k+1}$

$a_{k+2} = \sqrt{15 + 2a_{k+1}} > \sqrt{15 + 2a_k} = a_{k+1}$

Bounded above by 5 :

$n=1:\ a_1=3<5$

$n=k:\ a_{k+1}<5$

$n=k+1:\ a_{k+2}<5$

$a_{k+2}=\sqrt{15+2a_{k+1}}<\sqrt{15+2\,5}=5$

Since the sequence is increasing and bounded above it is convergent. Thus, the limit exist.

$\lim_{n\to\infty} a_n = x$ exists.

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{15+2a_n}=x$

$\sqrt{15+2x}=x \quad \Rightarrow x=5$

b) $\lim_{n\to\infty}(\sqrt{n^2+n}-\sqrt{n^2-1})\left(\dfrac{\sqrt{n^2+n}+\sqrt{n^2-1}}{\sqrt{n^2+n}+\sqrt{n^2-1}}\right)$

$=\lim_{n\to\infty} \dfrac{n^2+n-n^2+1}{n\sqrt{1+\dfrac{1}{n}}+n\sqrt{1-\dfrac{1}{n^2}}}=\dfrac{1}{2}$

The series is convergent.