KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 05.11.2015

İzmir University of Economics Faculty of Arts and Sciences, Department of Mathematics

Midterm Exam 1

Student Name and Surname:............................

Instructor's Name:...

(1) Evaluate the indicated limit. If it does not exist, is the limit ∞ or $-\infty$, or neither?

(a) $\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim_{x\to 2} \frac{(\sqrt{x^2+5})^2-(3)^2}{(x^2-2x)(\sqrt{x^2+5}+3)}$

$= \lim_{x\to 2} \frac{x^2-4}{(x^2-2x)(\sqrt{x^2+5}+3)} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x(x-2)(\sqrt{x^2+5}+3)}$

$= \lim_{x\to 2} \frac{x+2}{x(\sqrt{x^2+5}+3)}$

$= \frac{4}{2\cdot 6}$

$= \frac{1}{3}$

(b) $\lim_{x\to 5} \frac{x-7}{x(x-5)^3}$

$\lim_{x\to 5^-} \frac{x-7}{x(x-5)^3} = \infty$

$\lim_{x\to 5^+} \frac{x-7}{x(x-5)^3} = -\infty$

Since left and right limits are not equal, limit does not exist.

(c) $\lim_{x\to\infty} (\sqrt{x^6+5x^3}-x^3) \cdot \frac{(\sqrt{x^6+5x^3}+x^3)}{\sqrt{x^6+5x^3}+x^3}$

$= \lim_{x\to\infty} \frac{(\sqrt{x^6+5x^3})^2-(x^3)^2}{\sqrt{x^6+5x^3}+x^3} = \lim_{x\to\infty} \frac{5x^3}{\sqrt{x^6(1+\frac{5}{x^3})}+x^3}$

$= \lim_{x\to\infty} \frac{5x^3}{\underbrace{\sqrt{x^6}}_{=|x^3| = x^3 \text{ (as } x\to\infty)} \cdot \sqrt{1+\frac{5}{x^3}}+x^3} = \lim_{x\to\infty} \frac{5x^3}{\cancel{x^3}\left[\sqrt{1+\frac{5}{x^3}}+1\right]}$

$= \frac{5}{2}$

(2) Find the derivatives of the given functions.

(a) $y = \left(\frac{\sqrt{x}}{1+x}\right)^2$

$$y' = 2\left(\frac{\sqrt{x}}{1+x}\right)\cdot\left(\frac{\sqrt{x}}{1+x}\right)'$$

$$y' = 2\left(\frac{\sqrt{x}}{1+x}\right)\cdot\left(\frac{\frac{1}{2\sqrt{x}}\cdot(1+x) - \sqrt{x}\cdot 1}{(1+x)^2}\right)$$

$$y' = \frac{2\sqrt{x}}{1+x}\cdot\frac{\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2} - \sqrt{x}}{(1+x)^2}$$

$$y' = \frac{1-x}{(1+x)^3}$$

(b) $x^2 + y^2 = e^y + \cos(x^2)$

$$2x + 2y\cdot y' = e^y\cdot y' - \sin(x^2).\,2x$$

$$2yy' - e^y\cdot y' = -2x\cdot\sin(x^2) - 2x$$

$$y'(2y - e^y) = -2x\,(\sin(x^2)+1)$$

$$y' = \frac{-2x(\sin(x^2)+1)}{2y - e^y}$$

(c) $f(x) = x^{1-x}$

Apply natural logarithm to both sides:

$$\ln(f(x)) = \ln\left(x^{(1-x)}\right)$$

$$\ln(f(x)) = (1-x)\ln x$$

Then differentiate the above function:

$$\frac{1}{f(x)}\cdot f'(x) = -1.\ln x + (1-x).\frac{1}{x}$$

$$f'(x) = f(x)\left(-\ln(x) + \frac{1-x}{x}\right)$$

$$f'(x) = x^{1-x}.\left(-\ln x + \frac{1-x}{x}\right)$$

(3) Let $f(x) = \dfrac{x+1}{x-1}$.

(a) Find an equation of the tangent line to the graph of the given function at $(2, 3)$.

Slope of tangent line at $x=2$ is $f'(2)$

$$f'(x) = \frac{1.(x-1) - 1.(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$f'(2) = -2$

Equation of tangent line passing through $(2,3)$ with slope -2:

$$y-3 = -2.(x-2)$$

$$y-3 = -2x+4$$

$$y = -2x+7$$

(b) Find the points on the given function at which the tangent lines are perpendicular to the line $9x - 2y = 381$.

$9x-2y=381 \Rightarrow y = \frac{9}{2}x - \frac{381}{2}$ and slope is $\frac{9}{2}$

If tangent line is perpendicular to $9x-2y=381$, then

$$m_{tangent} \cdot \frac{9}{2} = -1 \Rightarrow m_{tangent} = -\frac{2}{9}$$

Let $x=a$ be the point of tangency. Then

$f'(a)$ is the slope of the tangent line which

must be $\frac{-2}{9}$. Hence,

$$f'(a) = \frac{-2}{(a-1)^2} = \frac{-2}{9} \Rightarrow \begin{array}{l} (a-1) = 3 \Rightarrow a=4 \\ a-1 = -3 \Rightarrow a=-2 \end{array}$$

$a=4 \Rightarrow f(4) = 5/3$

$a=-2 \Rightarrow f(-2) = 1/3$

$\therefore$ $(4, 5/3)$ and $(-2, 1/3)$ are the points

(4) (a) Given

$$f(x) = \begin{cases} 12 - 3x & ; x < 2 \\ 3 + a & ; x = 2 \\ x^2 - 2b & ; x > 2 \end{cases}$$

Find the values of the constants a and b such that f is continuous at $x = 2$.

If f is continuous at $x=2$, then

$$\lim_{x\to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (12-3x) = 6 \qquad \Rightarrow \quad 4-2b = 6 \Rightarrow \boxed{b=-1}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2-2b) = 4-2b$$

$$f(2) = 3+a$$

$$3+a = 6 \Rightarrow \boxed{a=3}$$

(b) Let $f(x) = \dfrac{x^3}{1+x^2}$. Show that f is one-to-one so that it has an inverse. Noting that $f(1/2) = 1$, find $(f^{-1})'(1/2)$.

$f(1) = 1/2$

$$f'(x) = \frac{3x^2(1+x^2) - 2x \cdot x^3}{(1+x^2)^2} = \frac{3x^2 + x^4}{(1+x^2)^2} > 0 \text{ for all } x \text{ so } f \text{ is}$$

increasing on its domain and one-to-one. So it is invertible.

The inverse of f is $x = \dfrac{y^3}{1+y^2}$ $\qquad f(1) = \frac{1}{2} \Rightarrow \boxed{f^{-1}(\tfrac{1}{2}) = 1}$

Differentiate $x = \dfrac{y^3}{1+y^2}$:

$$1 = \frac{3y^2 \cdot y' \cdot (1+y^2) - 2y \cdot y' \cdot y^3}{(1+y^2)^2} \Rightarrow 1 = \frac{y'(3y^2+y^4)}{(1+y^2)^2}$$

$$\Rightarrow y' = (f^{-1})'(x) = \left.\frac{(1+y^2)^2}{(3y^2+y^4)}\right|_{y=1} = \frac{4}{4} = 1$$