1. Evaluate the given limits

(a) $\lim_{x\to\infty} \frac{e^x+3}{x^2-5}$ $\left(\frac{\infty}{\infty}\right)$

Apply L'Hopital Rule

$= \lim_{x\to\infty} \frac{e^x}{2x}$ $\left(\frac{\infty}{\infty}\right)$

Apply L'Hopital Rule

$= \lim_{x\to\infty} \frac{e^x}{2}$

$= \infty$

(b) $\lim_{x\to\frac{\pi}{2}^-} \left(\frac{x}{\cot x} - \frac{\pi}{2\cos x}\right)$ $(\infty - \infty)$

$= \lim_{x\to\frac{\pi}{2}^-} \left(\frac{2x\sin x - \pi}{2\cos x}\right)$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to\frac{\pi}{2}^-} \frac{2\sin x + 2x\cos x}{-2\sin x}$

$= -1$

(c) $\lim_{x\to 0^+} (\sqrt{x})^x$

Let $y = (\sqrt{x})^x$, apply natural logarithm to both sides:

$\ln y = \ln(\sqrt{x})^x = x . \ln(\sqrt{x})$

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x . \ln(\sqrt{x})$ $(0 . \infty)$

$= \lim_{x\to 0^+} \frac{\ln(\sqrt{x})}{\frac{1}{x}}$

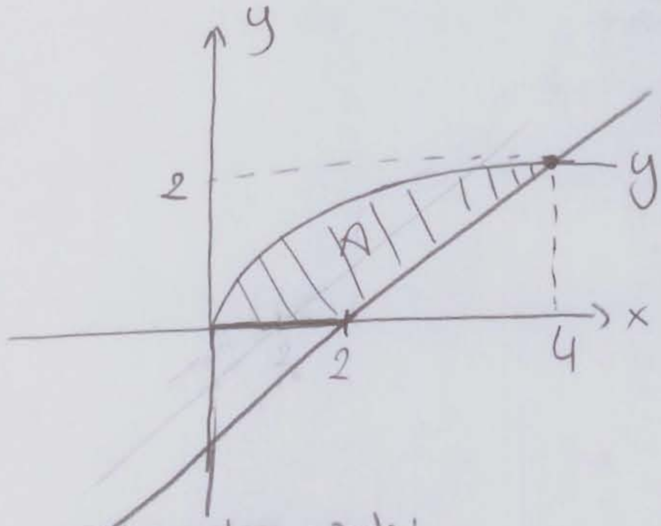
$= \lim_{x\to 0^+} \frac{\frac{1}{\sqrt{x}} . \frac{1}{2\sqrt{x}}}{\frac{-1}{x^2}}$

$= 0$

$\lim_{x\to 0^+} \ln y = \ln\left(\lim_{x\to 0^+} y\right) = 0$

$\therefore \lim_{x\to 0^+} \ln y = e^0 = 1$

2. (a) Sketch and find the area of the plane region bounded by $y = \sqrt{x}$, x-axis and $y = x - 2$.



$y = x - 2 \Rightarrow x = y + 2$

$y = \sqrt{x} \Rightarrow x = y^2$

$$A = \int_0^2 (y + 2 - y^2)\,dy$$

$$= \left(\frac{y^2}{2} + 2y - \frac{y^3}{3} \right) \Big|_0^2$$

$$= \frac{4}{2} + 4 - \frac{8}{3}$$

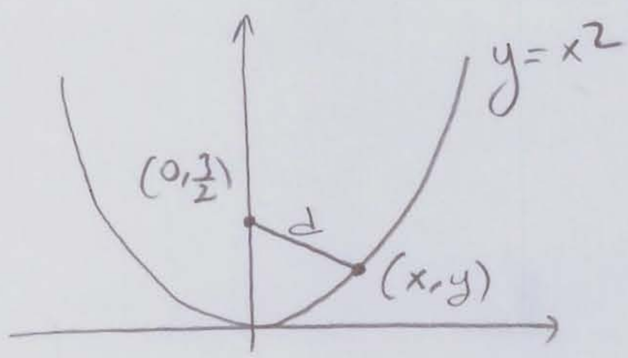
$$= \frac{10}{3}$$

Intersection points:

$x - 2 = \sqrt{x}$

$\Rightarrow x = 4$

(b) Find the shortest distance from the point $(0, \frac{3}{2})$ to the curve $y = x^2$.



$$\text{distance} = \sqrt{(x-0)^2 + \left(y - \tfrac{3}{2}\right)^2}$$

since $y = x^2$, distance can be written as:

$$\text{distance} = d(x) = \sqrt{x^2 + \left(x^2 - \tfrac{3}{2}\right)^2}$$

Take the derivative:

$$d'(x) = \frac{1}{2\sqrt{x^2 + \left(x^2 - \frac{3}{2}\right)^2}} \cdot \left(2x + 2\left(x^2 - \tfrac{3}{2}\right) \cdot 2x\right) = 0$$

$$\Rightarrow 2x + \cancel{2}\left(\frac{2x^2 - 3}{\cancel{2}}\right) \cdot 2x = 0$$

$$2x(1 + 2x^2 - 3) = 0 \quad \Rightarrow \quad x = 0,\ x = 1,\ x = -1$$

$$2x(2x^2 - 2) = 0$$

$x = 0 \Rightarrow y = 0$ and $d = \frac{3}{2}$

$x = 1 \Rightarrow y = 1$ and $d = \frac{\sqrt{5}}{2}$

$x = -1 \Rightarrow y = 1$ and $d = \frac{\sqrt{5}}{2}$

So, shortest distance is $\sqrt{5}/2$.

3. Let $f(x) = \dfrac{x^3}{3} - 2x^2 - 5x$.

(a) Find the domain, the x-intercept, the y-intercept and the asymptotes(if exist) of f.

Domain: $\mathbb{R}$

x-intercept:

$f(x) = \frac{x^3}{3} - 2x^2 - 5x = 0$

$x\left(\frac{x^2}{3} - 2x - 5\right) = 0$

$x = 0,\ x = 3 + \frac{\sqrt{96}}{2},\ x = 3 - \frac{\sqrt{96}}{2}$

y-intercept:

$x = 0 \Rightarrow f(0) = 0$

$(0,0)$

Asymptotes: none

(b) Find the intervals of increase and decrease and the local extreme value(s) of f.

$f'(x) = x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5,\ x = -1$

$f'(x)$: $+$ on $(-\infty,-1)$, $-$ on $(-1,5)$, $+$ on $(5,\infty)$

$f(-1) = \frac{8}{3}$ local max

$f(5) = -\frac{100}{3}$ local min

f is increasing on $(-\infty,-1) \cup (5,\infty)$

f is decreasing on $(-1,5)$

(c) Find the intervals where f is concave up and concave down, and inflection point(s).

$f''(x) = 2x - 4 = 0$

$x = 2$

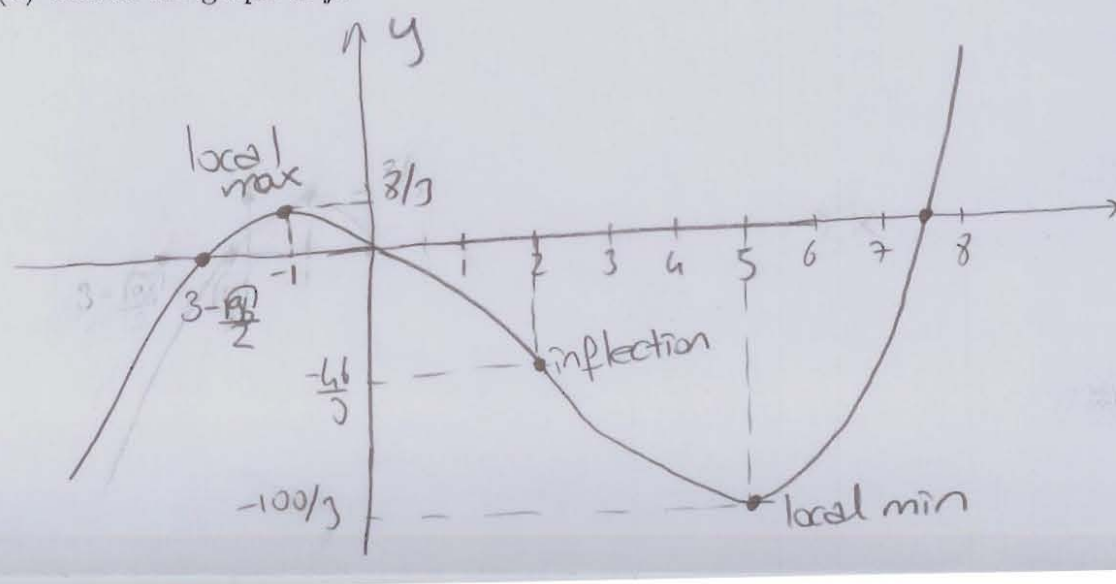
$f''(x)$: $-$ on $(-\infty,2)$, $+$ on $(2,\infty)$

f is concave up on $(2,\infty)$

f is concave down on $(-\infty,2)$

$f(2) = -\frac{46}{3}$ is inflection point

(d) Sketch the graph of f.



4. Evaluate the given integrals

(a) $\int \frac{x^3}{16-x^2}\,dx$ $= \int \left(-x + \frac{16x}{16-x^2}\right) dx = \int \left(-x + \frac{8}{4-x} - \frac{8}{4+x}\right) dx$

$$x^3 \div (16-x^2) = -x, \quad x^3 - (-16x + x^3) = 16x$$

$$= -\frac{x^2}{2} - 8\ln|4-x| - 8\ln|4+x| + C$$

$$\frac{16x}{(4-x)(4+x)} = \frac{A}{4-x} + \frac{B}{4+x} \Rightarrow 16x = A(4+x) + B(4-x)$$

$x = 4 \Rightarrow 64 = A \cdot 8 + B \cdot 0 \Rightarrow A = 8$

$x = -4 \Rightarrow -64 = A \cdot 0 + B \cdot 8 \Rightarrow B = -8$

(b) $\int \frac{\ln x}{x^5}\,dx$

$u = \ln x \qquad dv = \frac{1}{x^5}dx$

$du = \frac{1}{x}dx \qquad v = \frac{-1}{4x^4}$

$\int u\,dv = u \cdot v - \int v \cdot du$

$$= \frac{-\ln x}{4x^4} - \int \frac{-1}{4x^4} \cdot \frac{1}{x} \cdot dx$$

$$= \frac{-\ln x}{4x^4} + \frac{1}{4} \cdot \int x^{-5} dx$$

$$= \frac{-\ln x}{4x^4} - \frac{1}{16x^4} + C$$

(c) $\int_2^3 \frac{x}{(x-1)^{3/2}}\,dx$ $= \int_1^2 \frac{u+1}{u^{3/2}} \cdot du$

Let $u = x - 1$

$du = dx$

$x = u + 1$

$x = 2 \Rightarrow u = 1$

$x = 3 \Rightarrow u = 2$

$$= \int_1^2 \left(u^{-\frac{1}{2}} + u^{-\frac{3}{2}}\right) du$$

$$= \left(\frac{u^{1/2}}{1/2} + \frac{u^{-1/2}}{-1/2}\right)\Bigg|_1^2$$

$$= \left(2\sqrt{u} - \frac{2}{\sqrt{u}}\right)\Bigg|_1^2$$

$$= \left(2\sqrt{2} - \frac{2}{\sqrt{2}}\right) - (2-2)$$

$$= \frac{2}{\sqrt{2}}$$