KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 07.11.2013

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. Evaluate the given limits

(a) $\lim_{x\to 3} \frac{\sqrt{3x+16}-5}{x-3} \cdot \frac{\sqrt{3x+16}+5}{\sqrt{3x+16}+5}$

$= \lim_{x\to 3} \frac{(3x+16)-(5)^2}{(x-3)(\sqrt{3x+16}+5)}$

$= \lim_{x\to 3} \frac{3x-9}{(x-3)(\sqrt{3x+16}+5)}$

$= \lim_{x\to 3} \frac{3\cancel{(x-3)}}{\cancel{(x-3)}(\sqrt{3x+16}+5)}$

$= 3/10$

(b) $\lim_{x\to 1} \frac{x}{(x^2-1)^3} = \lim_{x\to 1} \frac{x}{(x-1)^3(x+1)^3}$

$\lim_{x\to 1^-} \frac{x}{(x^2-1)^3} = -\infty$

$\lim_{x\to 1^+} \frac{x}{(x^2-1)^3} = \infty$

	-1	0	1	
x	−	−	+	+
x−1	−	−	−	+
x+1	−	+	+	+
$x(x^2-1)^3$	−	+	−	+

Since left and right limits are not the same, limit does not exist.

(c) $\lim_{x\to 0} \frac{\sec x \tan x}{x}$

$= \lim_{x\to 0} \frac{\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}}{x}$

$= \lim_{x\to 0} \frac{1}{\cos^2 x} \cdot \frac{\sin x}{x}$

$= \underbrace{\lim_{x\to 0} \frac{1}{\cos^2 x}}_{=1} \cdot \underbrace{\lim_{x\to 0} \frac{\sin x}{x}}_{=1}$

$= 1$

2. (a) Find the derivative of $f(x) = 1 + \frac{1}{x^2}$ directly from the definition of derivative.

$$f(x) = 1 + \frac{1}{x^2}$$

$$f(x+h) = 1 + \frac{1}{(x+h)^2}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\left(1 + \frac{1}{(x+h)^2}\right) - \left(1 + \frac{1}{x^2}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{x^2 - (x+h)^2}{x^2 (x+h)^2 \cdot h} = \lim_{h \to 0} \frac{(x+x+h)(\cancel{x} - \cancel{x} - h)}{x^2 (x+h)^2 \cdot h}$$

$$= \lim_{h \to 0} \frac{(2x+h)(-\cancel{h})}{x^2 (x+h)^2 \cdot \cancel{h}} = \frac{-2x}{x^4}$$

$$= \frac{-2}{x^3}$$

(b) Find values of a and b that make

$$f(x) = \begin{cases} x + b & , \text{ if } x \le 1 \\ -ax + 3 & , \text{ if } x > 1 \end{cases}$$

differentiable at $x = 1$.

If f is differentiable then it is continuous.

f is continuous at $x = 1$: $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

$$\left.\begin{array}{l} \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x+b) = 1 + b \\ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (-ax+3) = -a + 3 \\ f(1) = 1 + b \end{array}\right\} \Rightarrow \begin{array}{l} 1 + b = -a + 3 \\ a + b = 2 \end{array}$$

f is differentiable at $x = 1$: $f'_+(1) = f'_-(1)$

$$\left.\begin{array}{l} f'_+(1) = (-ax+3)'\big|_{x=1} = -a \\ f'_-(1) = (x+b)'\big|_{x=1} = 1 \end{array}\right\} \Rightarrow -a = 1 \Rightarrow \boxed{a = -1} \Rightarrow \boxed{b = 3}$$

3. (a) Find the equation of the tangent to $y^3 + x^2 y = 2x + \ln y + 1$ at the point $(2, 1)$.

$$3y^2 \cdot y' + 2xy + x^2 y' = 2 + \frac{1}{y} \cdot y'$$

$$3y^2 y' + x^2 y' - \frac{1}{y} y' = 2 - 2xy$$

$$y'\left(3y^2 + x^2 - \frac{1}{y}\right) = 2 - 2xy$$

$$y' = \frac{2 - 2xy}{3y^2 + x^2 - \frac{1}{y}}$$

$$\begin{matrix} x = 2 \\ y = 1 \end{matrix} \Rightarrow \quad y'\Big|_{(2,1)} = \frac{2 - 2.2.1}{3.(1)^2 + (2)^2 - \frac{1}{1}} = \frac{-2}{6} = \frac{-1}{3}$$

Equation: $y - 1 = \frac{-1}{3}(x - 2)$

(b) Find all points on the curve $y = \dfrac{2}{x}$ where the tangent line is parallel to the line $y = \dfrac{-x}{2} + 1$.

Let $x = a$ and $y = \frac{2}{a}$ be the point of tangency

Then the slope of the tangent line is:

$$y' = \left(\frac{2}{x}\right)'\Big|_{x=a} = \frac{-2}{x^2}\Big|_{x=a} = \frac{-2}{a^2}$$

Since this tangent line is parallel to the line $y = \frac{-x}{2} + 1$, their slopes are equal.

slope of $y = \frac{-1}{2}x + 1$ is $-\frac{1}{2}$ so the slope of the tangent line is $\frac{-1}{2}$.

$$\frac{-2}{a^2} = \frac{-1}{2} \Rightarrow a^2 = 4 \Rightarrow a = \mp 2$$

$\boxed{\text{The points are } (2,1) \text{ and } (-2,-1)}$

4. (a) Find the derivative of $e^{xy}\cos\left(\frac{x}{y}\right) = x + \frac{1}{y}$

$$e^{xy}(xy)' \cdot \cos\left(\frac{x}{y}\right) + e^{xy} \cdot -\sin\left(\frac{x}{y}\right) \cdot \left(\frac{x}{y}\right)' = 1 - \frac{y'}{y^2}$$

$$e^{xy}(y + xy') \cdot \cos\left(\frac{x}{y}\right) - e^{xy}\sin\left(\frac{x}{y}\right)\left(\frac{y - xy'}{y^2}\right) = 1 - \frac{y'}{y^2}$$

(b) Find the derivative of $y = x^{\sin x}$

$$y = x^{\sin x}$$

$$\ln y = \ln x^{\sin x}$$

$$\ln y = \sin x \cdot \ln x$$

The derivative is:

$$\frac{1}{y} \cdot y' = \cos x \cdot \ln x + \sin x \cdot \frac{1}{x}$$

$$y' = y\left(\cos x \ln x + \frac{\sin x}{x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

(c) Suppose that $f(2) = 2$ and $f'(2) = 3$. Find $g'(2)$ if $g(x) = x^2 f(x)$

$$g(x) = x^2 f(x)$$

$$g'(x) = 2x f(x) + x^2 f'(x)$$

$$g'(2) = 2 \cdot 2 \cdot \underbrace{f(2)}_{=2} + (2)^2 \cdot \underbrace{f'(2)}_{=3}$$

$$g'(2) = 4 \cdot 2 + 4 \cdot 3$$

$$g'(2) = 20$$