KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 14.01.2014

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

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1. (a) Calculate enough derivatives of the function $f(x) = \ln(3x+5)$ to enable you to guess the general formula for the n-th derivative, $f^{(n)}(x)$. Then, verify your guess using mathematical induction.

$f(x) = \ln(3x+5)$

$f'(x) = \frac{1}{3x+5} \cdot 3 = 3 \cdot (3x+5)^{-1}$

$f''(x) = 3 \cdot (-1) \cdot (3x+5)^{-2} \cdot 3$

$f'''(x) = 3 \cdot (-1) \cdot (-2) \cdot (3x+5)^{-3} \cdot 3 \cdot 3$

$f^{IV}(x) = 3 \cdot (-1) \cdot (-2) \cdot (-3) \cdot (3x+5)^{-4} \cdot 3 \cdot 3 \cdot 3$

$\vdots$

$f^{(n)}(x) = 3^n \cdot (-1)^{n+1} \cdot (3x+5)^{-n} \cdot (n-1)!$

Substitute $k+1$ in place of n:

$f^{(k+1)}(x) = 3^{k+1} \cdot k! \cdot (-1)^{k+2} \cdot (3x+5)^{-(k+1)}$

i) For $n=1$:

$f^{(1)}(x) = 3^1 \cdot (-1)^2 \cdot (3x+5)^{-1}$

the formula is correct

ii) Assume that the formula is correct for $n=k$

$f^{(k)}(x) = 3^k \cdot (k-1)! \cdot (-1)^{k+1} \cdot (3x+5)^{-k}$

iii) Prove that it is correct for $n=k+1$:

$(f^{(k)})'(x) = \left[3^k \cdot (k-1)! (-1)^{k+1} (3x+5)^{-k}\right]'$

$f^{(k+1)}(x) = 3^k \cdot (k-1)! \cdot (-1)^{k+1} (-k) \cdot (3x+5)^{-k-1} \cdot 3$

$\Leftarrow = \Rightarrow \quad f^{(k+1)}(x) = 3^{k+1} \cdot k! \cdot (-1)^{k+2} \cdot (3x+5)^{-(k+1)}$

(b) The area of a circle increases at a rate of $1cm^2/s$. How fast is the radius changing when the circumference is $2cm$.

Area $= A = \pi r^2$

r: radius

Circumference : $C = 2\pi r$

$\frac{dA}{dt} = 1 cm^2/s$

$\frac{dr}{dt} = ?$

$C = 2cm$

$C = 2\pi r$ If $C = 2cm$ then $r = \frac{2}{2\pi} = \frac{1}{\pi} cm$

Differentiate area with respect to time:

$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$

$1 = \pi \cdot 2 \cdot \frac{1}{\pi} \cdot \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{2} cm/s$

2. Evaluate each improper integral or show that it diverges.

(a) $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{x^2+1}\,dx \;=\; \lim_{R\to\infty} \int_1^R \frac{\tan^{-1}x}{x^2+1}\,dx$

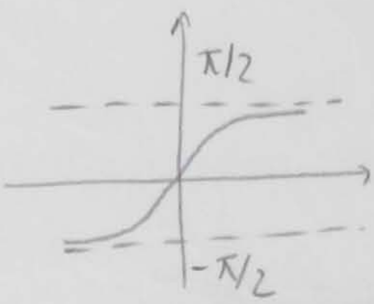
Let $u = \tan^{-1}x$

$du = \frac{1}{x^2+1}\,dx$

$$= \lim_{R\to\infty} \int u\,du$$

$$= \lim_{R\to\infty} \frac{u^2}{2}$$

$$= \lim_{R\to\infty} \left.\frac{(\tan^{-1}x)^2}{2}\right|_1^R$$



$$= \frac{1}{2}\lim_{R\to\infty}\left[\underbrace{(\tan^{-1}R)^2}_{\left(\frac{\pi}{2}\right)^2} - \underbrace{(\tan^{-1}1)^2}_{\left(\frac{\pi}{4}\right)^2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi^2}{4} - \frac{\pi^2}{16}\right] = \frac{3\pi^2}{32} \quad \text{converges.}$$

(b) $\displaystyle\int_2^3 \frac{1}{(x-2)^{1/3}}\,dx$

$$= \lim_{c\to 2^+} \int_c^3 \frac{1}{(x-2)^{1/3}}\,dx$$

$$= \lim_{c\to 2^+} \int_c^3 (x-2)^{-\frac{1}{3}}\,dx$$

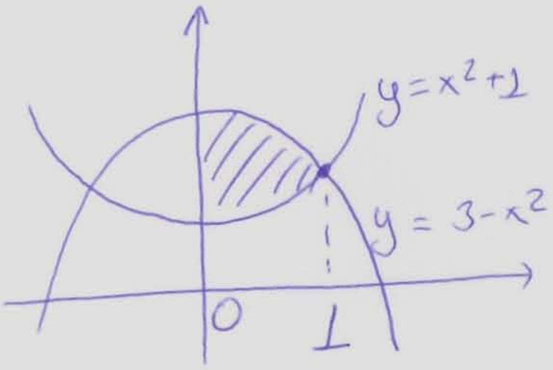
$$= \lim_{c\to 2^+} \left.\frac{(x-2)^{2/3}}{2/3}\right|_c^3$$

$$= \frac{3}{2}\lim_{c\to 2^+}\left[\underbrace{(3-2)^{2/3}}_{=1} - \underbrace{(c-2)^{2/3}}_{=0}\right]$$

$$= \frac{3}{2} \quad \text{converges}$$

3. Find the volume of the solid generated by rotating the region bounded by $y = x^2+1$, $y = 3 - x^2$, and $x \geq 0$ about

(a) the x-axis using slicing method



$$V = \int_0^1 \pi \left[(3-x^2)^2 - (x^2+1)^2 \right] dx$$

$$= \pi \int_0^1 \left[9 - 6x^2 + x^4 - x^4 - 2x^2 - 1 \right] dx$$

$$= \pi \int_0^1 (8 - 8x^2) dx$$

$$= 8\pi \left(x - \frac{x^3}{3} \right) \Big|_0^1$$

$$= \frac{16\pi}{3}$$

$3 - x^2 = x^2 + 1$

$x = 1$

$x = -1$

(b) the y-axis using cylindrical shell method.

$$V = 2\pi \int_0^1 x \left[(3-x^2) - (x^2+1) \right] dx$$

$$= 2\pi \int_0^1 x (2 - 2x^2) dx$$

$$= 4\pi \int_0^1 (x - x^3) dx$$

$$= 4\pi \left(\frac{x^2}{2} - \frac{x^4}{4} \right) \Big|_0^1$$

$$= \pi$$

4. Evaluate the given integrals

(a) $\int \frac{x}{e^{2x}}\,dx$ $= \int x e^{-2x} dx = x \cdot \frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} \cdot dx$

Let $u = x$

$du = dx$

$dv = e^{-2x} dx$

$v = \frac{e^{-2x}}{-2}$

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$= \frac{-x}{2e^{2x}} + \frac{1}{2} \cdot \frac{e^{-2x}}{-2} + c$

$= \frac{-x}{2e^{2x}} - \frac{1}{4e^{2x}} + c$

(b) $\int \frac{3x+2}{x^3 - 2x^2}\,dx$ $= \int \frac{3x+2}{x^2(x-2)} dx = \int \left(\frac{-2}{x} + \frac{-1}{x^2} + \frac{2}{x-2} \right) dx$

$\frac{3x+2}{x^2(x-2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2}$

$/x(x-2) \quad /x-2 \quad /x^2$

$3x+2 = A(x^2-2x) + B(x-2) + Cx^2$

$3x+2 = (A+C)x^2 + (-2A+B)x - 2B$

$-2B = 2 \Rightarrow B = -1$ | $-2A+B = 3 \Rightarrow A = -2$ | $A+C = 0$, $C = 2$

$= -2\ln|x| + \frac{1}{x} + 2\ln|x-2| + C$

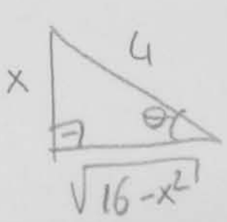
(c) $\int \frac{1}{(16 - x^2)^{3/2}}\,dx$ $= \int \frac{1}{4^3 \cos^3\theta} \cdot 4\cos\theta \cdot d\theta$

$x = 4\sin\theta \Rightarrow dx = 4\cos\theta\, d\theta$

$(16-x^2)^{3/2} = (16 - 16\sin^2\theta)^{3/2}$

$= (16\cos^2\theta)^{3/2}$

$= 4^3 \cos^3\theta$



$= \frac{1}{16} \int \frac{1}{\cos^2\theta} \cdot d\theta$

$= \frac{1}{16} \int \sec^2\theta\, d\theta$

$= \frac{1}{16} \tan\theta + C$

$= \frac{1}{16} \frac{x}{\sqrt{16-x^2}} + C$