KEY

25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 15.12.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

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1. (a) Evaluate the limit $\lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{2x}$

Let $y=\left(1+\frac{1}{3x}\right)^{2x}$. Apply $\ln$ to both sides

$\ln y=\ln\left(1+\frac{1}{3x}\right)^{2x} \Rightarrow \ln y = 2x.\ln\left(1+\frac{1}{3x}\right)$

$\lim_{x\to\infty}\ln y=\lim_{x\to\infty} 2x.\ln\left(1+\frac{1}{3x}\right)\quad(\infty.0)$

$=\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{3x}\right)}{\frac{1}{2x}}\quad\left(\frac{0}{0}\right)$

Apply L'Hopital Rule

$=\lim_{x\to\infty}\frac{\frac{1}{1+1/3x}\cdot\left(\frac{-1}{3x^2}\right)}{\frac{-1}{2x^2}}$

$=\frac{2}{3}$

$\lim_{x\to\infty}\ln\left(1+\frac{1}{3x}\right)^{2x}=\frac{2}{3}$

$\Rightarrow \lim_{x\to\infty}\left(1+\frac{1}{3x}\right)^{2x}=e^{\frac{2}{3}}$

(b) Let f be a differentiable function on $(-\infty,\infty)$ and $f'(1)\neq 0$. Evaluate the limit

$$\lim_{x\to 1}\frac{f(x)\sin\left(\frac{\pi x}{2}\right)-f(1)}{f(x)\cos(\pi x)+f(1)}$$

$\lim_{x\to 1}\frac{f(x).\sin\left(\frac{\pi x}{2}\right)-f(1)}{f(x).\cos(\pi x)+f(1)}\quad\left(\frac{0}{0}\right)$

Apply L'Hopital Rule

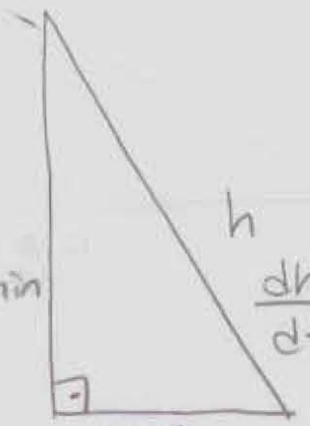
$=\lim_{x\to 1}\frac{f'(x).\sin\left(\frac{\pi x}{2}\right)+\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right).f(x)}{f'(x).\cos(\pi x)-\pi\sin(\pi x).f(x)}$

$=\frac{f'(1)\sin\left(\frac{\pi}{2}\right)+\frac{\pi}{2}\overbrace{\cos\left(\frac{\pi}{2}\right)}^{=0}.f(1)}{f'(1)\cos(\pi)-\pi\underbrace{\sin(\pi)}_{=0}.f(1)}$

$=\frac{\cancel{f'(1)}.\sin\left(\frac{\pi}{2}\right)}{\cancel{f'(1)}.\cos(\pi)}$

$=-1$

2. (a) One side of a right triangle $8cm$ long decreases at a rate of $1cm/min$ and the other side which is $6cm$ long increases at a rate of $2cm/min$. How fast is the hypothenuse changing?



$x = 8cm$

$\frac{dx}{dt} = -1 cm/min$

h

$\frac{dh}{dt} = ?$

$y = 6cm$

$\frac{dy}{dt} = 2cm/min$

$h^2 = x^2 + y^2 = 100$

$h = 10$

$h^2 = x^2 + y^2$

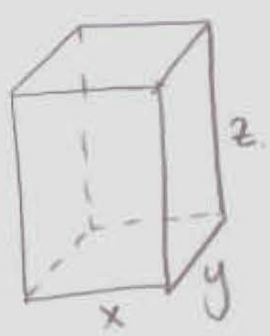
$\cancel{2}h \cdot \frac{dh}{dt} = \cancel{2}x\frac{dx}{dt} + \cancel{2}y\frac{dy}{dt}$

$10 \cdot \frac{dh}{dt} = 8 \cdot (-1) + 6 \cdot 2$

$\frac{dh}{dt} = \frac{4}{10} cm/min$

The hypothenuse is increasing at a rate of 0.4 cm/min

(b) Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width and height $64cm$. What are the dimensions and volume of a square-based box with the greatest volume under these conditions?



The box is square based so $x = y$

$x + y + z = 2x + z = 64$

Maximize $x \cdot y \cdot z = x^2 \cdot z$

If $2x + z = 64$ then $z = 64 - 2x$

Volume $= V(x) = x^2 \cdot (64 - 2x)$

$V'(x) = 2x \cdot (64 - 2x) - 2x^2 = 0$

$2x(64 - 2x - x) = 0$

$x \neq 0$, $\boxed{\frac{64}{3} = x}$

$x = \frac{64}{3} \Rightarrow y = \frac{64}{3}$ and $z = 64 - \frac{2 \cdot 64}{3} = \frac{64}{3}$

So, the greatest volume is: $V = \left(\frac{64}{3}\right)^3$ when $x = y = z = \frac{64}{3}$

3. Let $f(x) = \dfrac{x^2 - 2x + 1}{2x - 4}$.

(a) Find the domain, the intercepts, and the asymptotes (explain your answer) of f.

Domain: $\mathbb{R} - \{2\}$

x-intercept: $(f(x)=0)$

$\dfrac{x^2-2x+1}{2x-4} = 0 \Rightarrow x = 1$

$(1,0)$

y-intercept: $(x=0)$

$(0, -1/4)$

Vertical Asymptote: $x=2$

$\lim_{x\to 2^-} \dfrac{(x-1)^2}{2x-4} = -\infty$, $\lim_{x\to 2^+} \dfrac{(x-1)^2}{2x-4} = \infty$

Oblique Asymptote: $y = \frac{x}{2}$

$$\begin{array}{l|l} x^2-2x+1 & 2x-4 \\ \hline -x^2+2x & \frac{x}{2} \\ \hline 1 & \end{array}$$

(b) Find the intervals of increase and decrease and the local extreme value(s) (if any) of f.

$f(x) = \dfrac{x}{2} + \dfrac{1}{2x-4}$

$f'(x) = \dfrac{1}{2} - \dfrac{2}{(2x-4)^2}$

$f'(x) = \dfrac{(2x-4)^2 - 4}{2(2x-4)^2} = 0$

$x=3,\ x=1$ (critical points)

	$(-\infty,1)$	$(1,3)$	$(3,\infty)$
$f'(x)$	$+$	$-$	$+$
$f(x)$	inc.	dec.	inc.

$f(1) = 0$ local max value

$f(3) = 2$ local min value

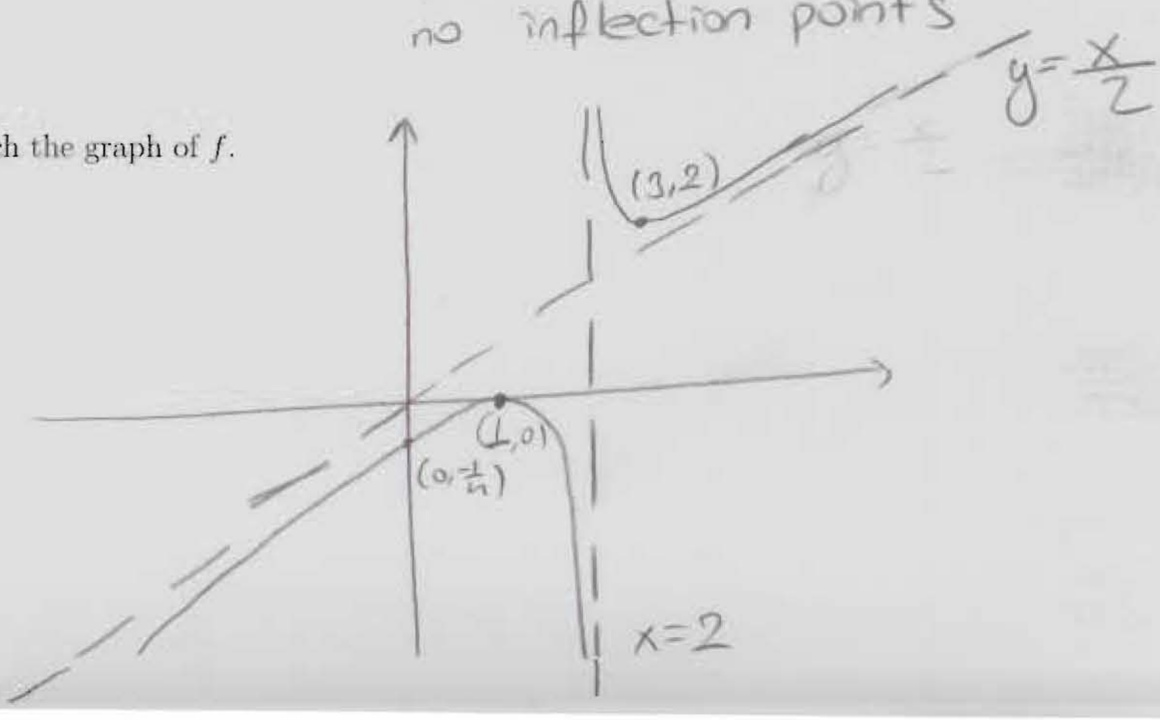
(c) Find the intervals where f is concave up and concave down, and inflection point(s)(if any).

$f''(x) = \dfrac{4}{(2x-4)^3}$

	$x<2$	$x>2$
$f''(x)$	$-$	$+$
the graph of $f(x)$	c.down	c.up

no inflection points

(d) Sketch the graph of f.



4. (a) Use a suitable linearization (linear approximation) to approximate $(2.02)^3$.

Find linearization of $f(x)=x^3$ about 2:

$$L(x) = f(2) + f'(2)\cdot(x-2)$$

$$\left\{\begin{array}{ll} f(x) = x^3 & \Rightarrow f(2) = 8 \\ f'(x) = 3x^2 & \Rightarrow f'(2) = 12 \end{array}\right\}$$

$$L(x) = 8 + 12(x-2)$$

$$(2.02)^3 = f(2.02) \approx L(2.02) = 8 + 12\cdot(2.02-2)$$
$$= 8 + 12\cdot(0.02)$$
$$= 8.24$$

So $(2.02)^3 \approx 8.24$

(b) Find the Maclaurin polynomial of order n for $f(x) = xe^{-x}$.

$$P_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(x) = xe^{-x}$ $\Rightarrow f(0) = 0$

$f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$ $\Rightarrow f'(0) = 1$

$f''(x) = -e^{-x}(1-x) - e^{-x} = -e^{-x}(2-x)$ $\Rightarrow f''(0) = -2$

$f'''(x) = e^{-x}(2-x) + e^{-x} = e^{-x}(3-x)$ $\Rightarrow f'''(0) = 3$

$f^{IV}(x) = -e^{-x}(3-x) - e^{-x} = -e^{-x}(4-x)$ $\Rightarrow f^{IV}(0) = -4$

$\vdots$

$f^{(n)}(x) = (-1)^{n+1}e^{-x}(n-x)$ $\Rightarrow f^{(n)}(0) = (-1)^{n+1}\cdot n$

$$P_n(x) = x - \frac{2}{2!}x^2 + \frac{3}{3!}x^3 - \frac{4}{4!}x^4 + \cdots + \frac{(-1)^{n+1}\cdot n}{n!}\cdot x^n$$

$$P_n(x) = \sum_{i=1}^{n} \frac{(-1)^{i+1}\cdot i}{i!}\cdot x^i$$