1. Let $f(x) = \frac{x^2-2}{x^2-1}$.

a) Find the domain, the intercepts, and the asymptotes of $f(x)$.

Domain $= (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

Intercepts $= (0, 2), (-\sqrt{2}, 0), (\sqrt{2}, 0)$

$\lim_{x \to +\infty} f(x) = 1$, $\lim_{x \to -\infty} f(x) = 1 \Rightarrow y = 1$ horizontal asymptote

$$\left.\begin{array}{l} \lim_{x \to 1^+} f(x) = -\infty, \quad \lim_{x \to 1^-} f(x) = +\infty \\ \lim_{x \to -1^+} f(x) = +\infty, \quad \lim_{x \to -1^-} f(x) = -\infty \end{array}\right\} \Rightarrow x = \mp 1 \text{ vertical asymptote}$$

b) Find the local extrema (if any) and determine the intervals where the function increases or decreases.

$$f'(x) = \frac{2x(x^2-1) - (x^2-2) \cdot 2x}{(x^2-1)^2} = \frac{2x}{(x^2-1)^2} = 0 \Rightarrow x = 0 \text{ critical point}$$

f' is undefined when $x = \mp 1$.

x		VA -1		CP 0		VA 1	
f'	$-$	undef	$-$	0	$+$	undef	$+$
f	↘		↘	loc min	↗		↗

c) Find the inflection points (if any) and determine the intervals of concavity.

$$f''(x) = \frac{-2(1+3x^2)}{(x^2-1)^3}$$

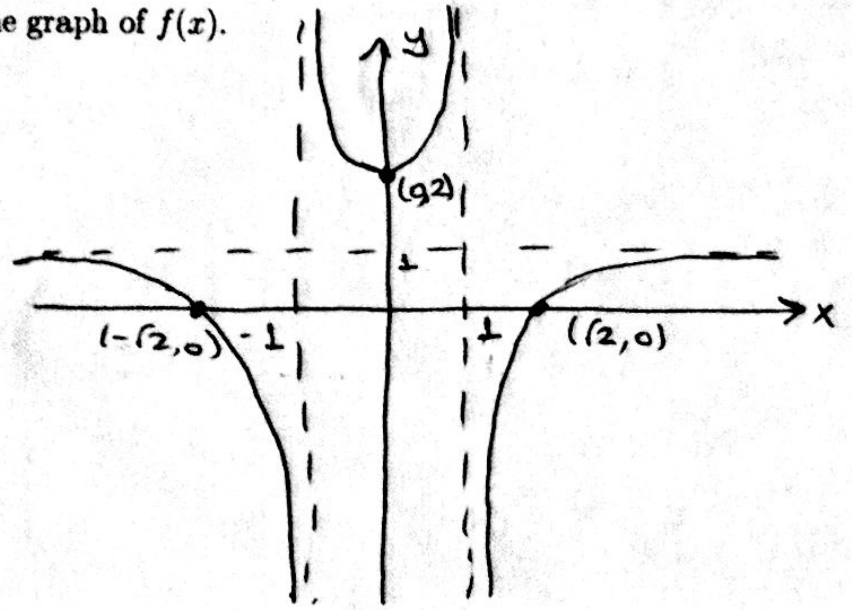
f'' is undefined when $x = \mp 1$.

x		VA -1		VA 1	
f''	$-$	undef	$+$	undef	$-$
f	$\cap$		$\cup$		$\cap$

Inflection points: none

$f''(x) = 0$ nowhere

d) Sketch the graph of $f(x)$.



2. (a) Evaluate the limit: $\lim_{x\to\infty} x(2\tan^{-1}x - \pi)$. $\infty.0$

$$= \lim_{x\to\infty} \frac{2\tan^{-1}x - \pi}{\frac{1}{x}} \quad \frac{0}{0}$$

$$= \lim_{x\to\infty} \frac{2.\frac{1}{1+x^2}}{\frac{-1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-2x^2}{1+x^2}$$

$$= \lim_{x\to\infty} \frac{-2}{\underbrace{\frac{1}{x^2}}_{0}+1}$$

$$= -2$$

(b) Evaluate the limit: $\lim_{x\to 0^+} x^{\sqrt{x}}$.

Let $y = x^{\sqrt{x}}$. Apply natural log. to both sides

$\ln y = \ln x^{\sqrt{x}} \Rightarrow \ln y = \sqrt{x}.\ln x$

Evaluate $\lim_{x\to 0^+} \sqrt{x}.\ln x$ $(0.\infty)$

$$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sqrt{x}}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{2x^{3/2}}}$$

$$= \lim_{x\to 0^+} \frac{-2x^{3/2}}{x}$$

$$= 0$$

Since $\lim_{x\to 0^+} \ln y = 0$, $\lim_{x\to 0^+} y = \lim_{x\to 0^+} x^{\sqrt{x}} = e^0 = 1$

3. (a) Use a suitable linearization to find an approximate value of $\sin 33°$.

$f(x) = \sin x$, $f'(x) = \cos x$

about a

$L(x) = f(a) + f'(a).(x-a)$

$\sin 33° = \sin(30° + 3°)$

$\sin 33° = \sin\left(\frac{\pi}{6} + \frac{\pi}{60}\right)$

$L(x) = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right).\left(x - \frac{\pi}{6}\right)$

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2}$

$f'\left(\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$L\left(\frac{\pi}{6} + \frac{\pi}{60}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\left(\cancel{\frac{\pi}{6}} + \frac{\pi}{60} - \cancel{\frac{\pi}{6}}\right)$

$= \frac{1}{2} + \frac{\sqrt{3}}{2}.\frac{\pi}{60} = \frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\left(\frac{\pi}{60}\right) \approx 0.545$

(b) Find the nth order Maclaurin polynomial P_n for $\frac{1}{1-x}$.

$f(x) = \frac{1}{1-x}$

$P_n(0) = f(0) + \frac{f'(0)}{1!}.(x-0) + \frac{f''(0)}{2!}.(x-0)^2 + \frac{f'''(0)}{3!}.(x-0)$

$+ \dots + \frac{f^{(n)}(0)}{n!}.(x-0)^n$

$f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$

$f''(x) = \frac{2.(1-x)}{(1-x)^4} = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 1.2$

$f'''(x) = \frac{2.3.(1-x)^2}{(1-x)^6} = \frac{2.3}{(1-x)^4} \Rightarrow f'''(0) = 1.2.3$

$\vdots$

$f^{(n)}(x) = \frac{n!}{(1-x)^{(n+1)}} = n! \Rightarrow f^{(n)}(0) = n!$

$P_n(x) = 1 + \frac{1}{1!}x + \frac{1.2}{2!}.x^2 + \frac{1.2.3}{3!}.x^3 + \dots + \frac{1.2\dots n}{n!}.x^n$

$P_n(x) = 1 + x + x^2 + x^3 + \dots + x^n$

4. (a) The area of a rectangle is increasing at rate of $5m^2/s$ while the length is increasing at a rate of $10m/s$. If the length is $20m$ and the width is $16m$, how fast is the width changing?

Length $= x$, Width $= y$, Area $= A = xy$ and

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt}. \text{ If } \frac{dA}{dt} = 5, \frac{dx}{dt} = 10, x = 20, y = 16$$

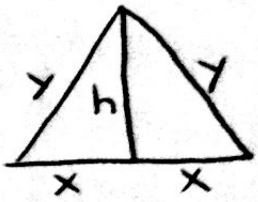
Then

$$5 = 20\frac{dy}{dt} + 16(10) \Rightarrow \frac{dy}{dt} = -\frac{31}{4}.$$

Thus, the width is decreasing at $\frac{31}{4}$ m/s

(b) Among all isosceles triangles of given perimeter, show that the equilateral triangle has the greatest area.

Let the dimensions be as shown:



Then $2x + 2y = P$ (given constant)

The area is:

$$A = xh = x\sqrt{y^2 - x^2} = x\sqrt{\left(\frac{P}{2} - x\right)^2 - x^2}.$$

$y \geq x$, so $0 \leq x \leq P/4$. If $x = 0$ or $x = P/4$, then $A = 0$. So max occurs at CP.

$$0 = \frac{dA}{dx} = \sqrt{\frac{P^2}{4} - Px} - \frac{Px}{2\cdot\sqrt{\frac{P^2}{4} - Px}}.$$

$\frac{P^2}{2} - 2Px - Px = 0$ or $x = \frac{P}{6}$. Thus $y = P/3$ and the triangle is equilateral, all sides are $P/3$.