25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 12.12.2009

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

SECOND MIDTERM EXAM

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Duration: 110 mins

1. Sketch the graph of the function $f(x) = \dfrac{x}{x^2 - 4}$ by finding the domain, intercepts, asymptotes, intervals where function is incerasing/decreasing, if any, the inflection point(s), and the intervals of concavity.

Solution :

Step1. Domain of the function is $\mathcal{D}(f) = \mathbf{R} - \{-2, 2\}$.

Since $x^2 \neq 4$.

$f(x)$ has vertical asymptotes at $x = -2$ and $x = 2$.

Since $\lim_{x\to -2^-} f(x) = -\infty$, and $\lim_{x\to -2^+} f(x) = \infty$, Also, $\lim_{x\to 2^-} f(x) = -\infty$, and $\lim_{x\to 2^+} f(x) = \infty$,

$f(x)$ has horizontal asymptote at $y = 0$.

Since $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 0$,

$f(x)$ has symmetry about the origin, since $f(x)$ is odd function that is $f(-x) = f(x)$.

Intercepts: $x = 0 \quad \Rightarrow \quad y = 0$.

Step2. $f'(x) = \frac{-(x^2+4)}{(x^2-4)^2}$ so $f(x)$ has no critical point.

Since $x^2 + 4 \neq 0$.

And $f(x)$ has no singular point.

Since $x^2 = 4 \Rightarrow x = 2$ and $x = -2$. But $-2 \notin \mathcal{D}(f)$ and $2 \notin \mathcal{D}(f)$.

$f'(x) < 0$ for all $x \in \mathbf{R} - \{-2, 2\}$. So $f(x)$ is decreasing for all x except $x = -2$ and $x = 2$.

And $f(x)$ has no local extrema point.

Step3. $f''(x) = \frac{2x(x^2+12)}{(x^2-4)^3}$.

Also $f''(x) > 0$ on $(-2, 0) \cup (2, \infty)$ and $f''(x) < 0$ on $(-\infty, -2) \cup (0, 2)$.

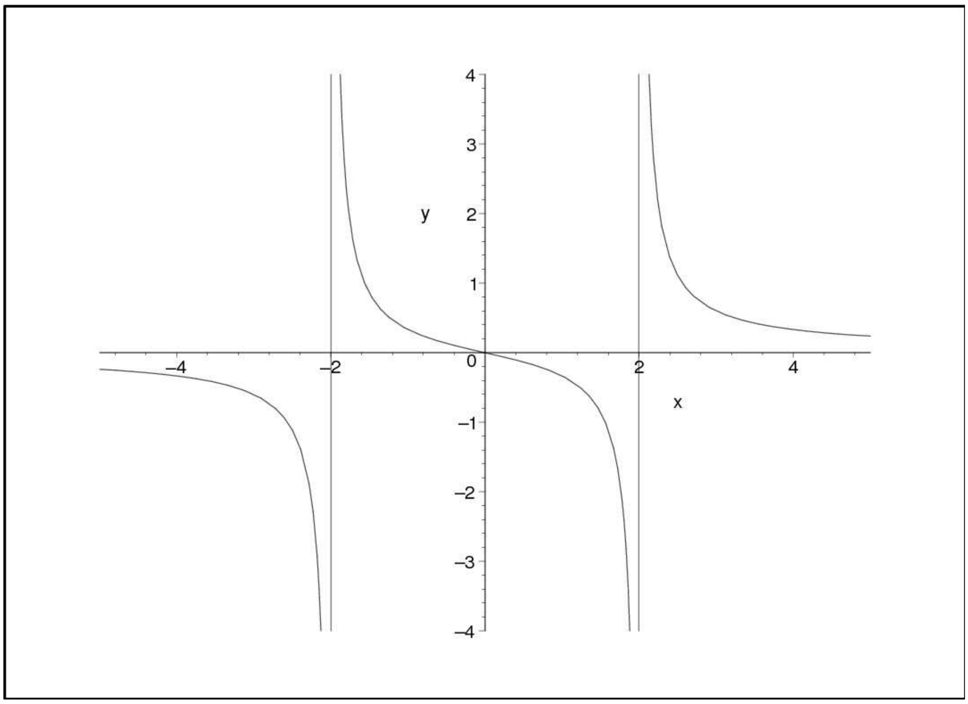
So, $f(x)$ is concave up on $(-2, 0) \cup (2, \infty)$ and $f(x)$ is concave down on $(-\infty, -2) \cup (0, 2)$.

If $f''(x) = 0$ then $x = 0$. Then $x = 0$ is inflection point.

Since $0 \in \mathcal{D}(f)$ and $f''(x)$ changes its sign from positive to negative.

Although, $f''(x)$ changes its sign at $x = -2$ and $x = 2$. But $x = -2$ and $x = 2$ are not inflection points. Since $-2 \notin \mathcal{D}(f)$ and $2 \notin \mathcal{D}(f)$.

Hence, the graph of $f(x)$ follows:



2. (a) Show that $f(x) = 3x + x^3$ is one-to-one on the whole real line and find $\left(f^{-1}\right)'(4)$.

Solution :

The function $f(x)$ is one-to-one function. Since $f'(x) = 3 + 3x^2 > 0$ for all $x \in \mathbf{R}$ that means $f(x)$ is increasing function for all $x \in \mathbf{R}$.

$[f^{-1}(4)]' = \dfrac{1}{f'(y)\Big|_{y=f^{-1}(4)}}$. And $f^{-1}(4) = x \Rightarrow f(x) = 4$.

That means $x^3 + 3x = 4 \Rightarrow x = 1$. So, $f^{-1}(4) = 1$. And

$[f^{-1}(4)]' = \dfrac{1}{f'(y)\Big|_{y=1}} = \dfrac{1}{(3+3y^2)\Big|_{y=1}} = \dfrac{1}{6}$.

(b) Solve $4\ln\left(\sqrt{x}\right) + 6\ln x^{1/3} = 4$.

Solution :

$$\begin{aligned}
4 &= 4\ln\left(\sqrt{x}\right) + 6\ln x^{1/3} \\
&= \ln\left(\sqrt{x}\right)^4 + \ln(x^{1/3})^6 \\
&= \ln x^2 + \ln x^2 \\
&= \ln[(x^2).(x^2)] \\
&= \ln x^4 \\
&= 4\ln x
\end{aligned}$$

Hence, $\ln x = 1 \Rightarrow x = e$.

3. (a) Evaluate $\lim_{x\to 0}(\cos 2x)^{(1/x^2)}$

Solution :

Let $y = (\cos 2x)^{(1/x^2)}$, then

$$\lim_{x\to 0} y = \lim_{x\to 0}(\cos 2x)^{(1/x^2)}$$

And $\ln y = \ln\left((\cos 2x)^{(1/x^2)}\right)$, then

$$\begin{aligned} \lim_{x\to 0}\ln y &= \lim_{x\to 0}\ln\left((\cos 2x)^{(1/x^2)}\right) \\ &= \lim_{x\to 0}\frac{1}{x^2}\ln(\cos 2x) \qquad \left[\frac{0}{0}\right] \end{aligned}$$

Then by applying L'Hospital rule, since $x^2 \neq 0$ for any $x \neq 0$. Then

$$\lim_{x\to 0}\frac{[\ln(\cos 2x)]'}{[x^2]'} = \lim_{x\to 0}\frac{\frac{-2\sin 2x}{\cos 2x}}{2x} = \lim_{x\to 0}\frac{-2\sin 2x}{2x\cos 2x} \qquad \left[\frac{0}{0}\right]$$

By applying L'Hospital rule again, we have

$$\lim_{x\to 0}\frac{[-2\sin 2x]'}{[2x\cos 2x]'} = \lim_{x\to 0}\frac{-2\sec^2 2x}{1} = -2$$

Then,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{1}{x^2}\ln(\cos 2x) = -2$$

$$\lim_{x\to 0} e^{\ln y} = \lim_{x\to 0} e^{\frac{1}{x^2}\ln(\cos 2x)} = e^{-2}$$

Hence,

$$\lim_{x\to 0} y = e^{-2}.$$

(b) Evaluate $\lim_{x\to -\infty}\frac{x^2}{e^{-x}}$

Solution :

$$\lim_{x\to -\infty}\frac{x^2}{e^{-x}} \qquad \left[\frac{\infty}{\infty}\right]$$

Then by applying L'hospital rule, since $e^{-x} \neq 0$ for any $x \in \mathbf{R}$. Then

$$\lim_{x\to -\infty}\frac{[x^2]'}{[e^{-x}]'} = \lim_{x\to -\infty}\frac{2x}{-e^{-x}} \qquad \left[\frac{-\infty}{\infty}\right]$$

By applying L'Hospital rule again, we have

$$\lim_{x\to -\infty}\frac{[2x]'}{[-e^{-x}]'} = \lim_{x\to -\infty}\frac{2}{e^{-x}} = 0$$

Hence,

$$\lim_{x\to -\infty}\frac{x^2}{e^{-x}} = 0.$$

4. (a) Differentiate the function $f(x) = \ln(x^2+4) - x\tan^{-1}(\sqrt{x^2+1})$.

Solution :

$$f(x) = \ln(x^2+4) - x\tan^{-1}(\sqrt{x^2+1})$$

Differentiating $f(x)$ with respect to x yields

$$f'(x) = \frac{2x}{(x^2+4)} - \left(\tan^{-1}(\sqrt{x^2+1}) + x\frac{1}{1+(\sqrt{x^2+1})^2}\frac{2x}{2\sqrt{x^2+1}}\right)$$

$$f'(x) = \frac{2x}{(x^2+4)} - \tan^{-1}(\sqrt{x^2+1}) - \frac{x^2}{(x^2+2)(\sqrt{x^2+1})}$$

(b) Two numbers have sum 16. What are the numbers if the product of the cube of one and fifth power of the other is as large as possible?

Solution :

Let the numbers be x and y such as $x+y=16$. It is clear that $x \geq 0$ and $y \geq 0$ must be satisfied. Let

$$P = x^3y^5$$

If $y = 16 - x$ is substituted instead of y, then we write P in terms of one variable only

$$P(x) = x^3(16-x)^5$$

Here $0 \leq x \leq 16$, the maximum value must occur at critical points, end points or singular points.

$$P'(x) = x^2(16-x)^4(48-8x)$$

Since $P'(x)$ is defined for all $x \in \mathbf{R}$, there is no singular point.

Critical points are $x=0$, $x=6$, $x=16$. End points are $x=0$, $x=16$.

$P(0) = 0$

$P(16) = 0$

$P(6) = 216 \times 10^5$. Thus $P(x)$ is the maximum if $x=6$ and $y=10$.