| 25 points | 25 points | 25 points | 25 points | 100 points |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | **Total** |
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# MATH 153 CALCULUS I 09.11.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

## FIRST MIDTERM EXAM

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1. Evaluate the given limits

(a) $\lim_{x\to 0} \frac{x}{|x-1|-|x+1|}$

$|x-1| = \begin{cases} x-1 & ; x \geq 1 \\ \underline{-x+1} & ; x < 1 \end{cases}$ $\quad |x+1| = \begin{cases} \underline{x+1} & ; x \geq -1 \\ -x-1 & ; x < -1 \end{cases}$

$$\lim_{x\to 0} \frac{x}{-x+1-(x+1)} = \lim_{x\to 0} \frac{x}{-2x} = \frac{-1}{2}$$

(b) $\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right)$

Use squeeze Theorem

$$-1 \leq \sin\left(\frac{\pi}{x}\right) \leq 1$$

$$-\sqrt{x^3+x^2} \leq \sqrt{x^3+x^2}\cdot\sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3+x^2}$$

$$\underbrace{\lim_{x\to 0}\left(-\sqrt{x^3+x^2}\right)}_{=0} \leq \lim_{x\to 0} \sqrt{x^3+x^2}\sin\left(\frac{\pi}{x}\right) \leq \underbrace{\lim_{x\to 0}\left(\sqrt{x^3+x^2}\right)}_{=0}$$

$$\therefore \lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) = 0$$

(c) $\lim_{x\to 0} \frac{x}{\sqrt{1+\tan(2x)}-1}$

$$= \lim_{x\to 0} \frac{x}{\sqrt{1+\tan(2x)}-1} \cdot \frac{\sqrt{1+\tan(2x)}+1}{\sqrt{1+\tan(2x)}+1}$$

$$= \lim_{x\to 0} \frac{x\left(\sqrt{1+\tan(2x)}+1\right)}{\tan(2x)}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+\tan(2x)}+1}{\frac{\sin(2x)}{x}\cdot\frac{1}{\cos(2x)}}$$

$$= \lim_{x\to 0} \frac{\sqrt{1+\tan(2x)}+1}{\frac{\sin(2x)}{2x}\cdot\frac{2}{\cos(2x)}}$$

$$= \lim_{x\to 0} \frac{1}{\underbrace{\frac{\sin(2x)}{2x}}_{=1}} \cdot \underbrace{\frac{\sqrt{1+\tan(2x)}+1}{\frac{2}{\cos(2x)}}}_{=1}$$

$$= 1$$

2. (a) Let $f$ be a differentiable function and $g(x) = f(f(x^3) + x)$. If $f(1) = 1$, $f'(1) = -3$, and $g'(1) = 2$ find $f'(2)$.

$$g(x) = f(f(x^3)+x)$$

$$g'(x) = f'(f(x^3)+x) \cdot [f(x^3)+x]'$$

$$g'(x) = f'(f(x^3)+x) \cdot [f'(x^3) \cdot 3x^2 + 1]$$

$x=1 \Rightarrow g'(1)=2,\ f(1)=1$ and $f'(1)=-3$

$$\underbrace{g'(1)}_{=2} = f'(\underbrace{f(1)}_{=1}+1) \cdot [\underbrace{f'(1)}_{=-3} \cdot 3+1]$$

$$2 = f'(2) \cdot (-8)$$

$$f'(2) = \frac{-1}{4}$$

(b) Find values of $a$ and $b$ that make

$$f(x) = \begin{cases} \sin(ax) + b\ ;\ x < 0 \\ \sin^2(2x) + 2x\ ;\ x \geq 0 \end{cases}$$

differentiable at $x = 0$. (Hint: Check the continuity.)

If $f$ is differentiable then it is continuous. So,

$$\lim_{x \to 0} f(x) = f(0)$$

$$\left.\begin{array}{l} \lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (\sin(ax)+b) = b \\ \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (\sin^2(2x) + 2x) = 0 \\ f(0) = 0 \end{array}\right\} \Rightarrow b = 0$$

If $f$ is differentiable, at $x=0$, then $f'_+(0) = f'_-(0)$

$$\left.\begin{array}{l} f'_+(0) = (2\sin(2x)\cos(2x) \cdot 2 + 2)\Big|_{x=0} = 2 \\ f'_-(0) = (\cos(ax) \cdot a)\Big|_{x=0} = a \end{array}\right\} \Rightarrow a = 2$$

**3.** (a) Does the graph of the function $f(x) = (x+1)^{2/3}$ have a tangent line at $x = -1$? If yes, what is the tangent line?

If $\lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h}$ exists, then f has a tangent line at $x=-1$.

$f(x) = (x+1)^{2/3} \Rightarrow f(-1)=0$ and $f(-1+h) = h^{2/3}$

$$\lim_{h\to 0} \frac{f(-1+h)-f(-1)}{h} = \lim_{h\to 0} \frac{h^{2/3}-0}{h} = \lim_{h\to 0} \frac{1}{h^{1/3}} = ?$$

$$\lim_{h\to 0^-} \frac{1}{h^{1/3}} = -\infty \quad \text{and} \quad \lim_{h\to 0^+} \frac{1}{h^{1/3}} = \infty$$

Thus, limit does not exist and f does not have a tangent line at $x=-1$.

(b) Find the equation of a straight line that passes through the point $(1, 0)$ and is tangent to the curve $y = \dfrac{x}{x-1}$.

Assume that the line is tangent to the curve at $\left(a, \frac{a}{a-1}\right)$. Then, the slope is:

$$m = y'(a) = \left(\frac{x}{x-1}\right)'\Big|_{x=a} = \frac{-1}{(x-1)^2}\Big|_{x=a} = \frac{-1}{(a-1)^2}$$

This line is passing through the points $(1,0)$ and $\left(a, \frac{a}{a-1}\right)$. Then, the slope is:

$$m = \frac{\frac{a}{a-1} - 0}{a-1} = \frac{a}{(a-1)^2}$$

Equating these slopes yields:

$$\frac{-1}{(a-1)^2} = \frac{a}{(a-1)^2} \Rightarrow a = -1$$

Equation of line:

$\left(-1, \frac{1}{2}\right), m = \frac{-1}{4}$

$y - \frac{1}{2} = \frac{-1}{4}(x-(-1))$

$y = \frac{-1}{4}x + \frac{1}{4}$

4. (a) Solve the given second order initial value problem

$$\begin{cases} y''(x) = x^{-3} - x \\ y'(1) = 2 \\ y(1) = -1 \end{cases}$$

Where the solution is valid?

$y'(x) = \int (x^{-3} - x)dx$

$y'(x) = \frac{x^{-2}}{-2} - \frac{x^2}{2} + c_1$

$x = 1 \Rightarrow y'(1) = 2$ :

$2 = \frac{-1}{2} - \frac{1}{2} + c_1$

$c_1 = 3$

$y'(x) = \frac{-x^{-2}}{2} - \frac{x^2}{2} + 3$

$y(x) = \int \left(\frac{-x^{-2}}{2} - \frac{x^2}{2} + 3\right) dx$

$y(x) = \frac{x^{-1}}{2} - \frac{x^3}{6} + 3x + c_2$

$x = 1 \Rightarrow y(1) = -1$ :

$-1 = \frac{1}{2} - \frac{1}{6} + 3 + c_2$

$c_2 = \frac{-13}{3}$

$\therefore y(x) = \frac{1}{2x} - \frac{x^3}{6} + 3x - \frac{13}{3}$

The solution is valid on $(0, \infty)$.

(b) Find the slope of the curve $x \sin(xy - y^2) = x^2 - 1$ at $(1, 1)$.

$\frac{d}{dx}\left[x \sin(xy - y^2)\right] = \frac{d}{dx}(x^2 - 1)$

$\sin(xy - y^2) + x\cos(xy - y^2) \cdot (y + xy' - 2yy') = 2x$

$x = 1,\ y = 1$ :

$\sin(0) + 1 \cdot \cos(0) \cdot (1 + y' - 2y') = 2$

$1 - y' = 2$

$y' = -1$