25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 15.11.2011

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. Evaluate the limit or explain why it does not exist.

(a) $\lim_{x \to 3} \frac{\sqrt{9 - 6x + x^2}}{x - 3}$

$$\lim_{x \to 3} \frac{\sqrt{x^2 - 6x + 9}}{x - 3} = \lim_{x \to 3} \frac{\sqrt{(x-3)^2}}{x-3} = \lim_{x \to 3} \frac{|x-3|}{x-3}$$

$$\lim_{x \to 3-} \frac{-(x-3)}{x-3} = -1 \neq \lim_{x \to 3+} \frac{x-3}{x-3} = 1 \Rightarrow \text{The limit does not exist.}$$

(b) $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right)$

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1 \Rightarrow -x^2 \leq x^2 \cos\left(\frac{1}{x}\right) \leq x^2$$

By the Squzee Theorem since $\lim_{x \to 0} x^2 = \lim_{x \to 0} (-x^2) = 0$, we get

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$$

(c) $\lim_{x \to \infty} \frac{1}{\sqrt{x^2 - 2x} - x}$

$$\lim_{x \to \infty} \frac{\sqrt{x^2 - 2x} + x}{(\sqrt{x^2 - 2x} - x)(\sqrt{x^2 - 2x} + x)} = \lim_{x \to \infty} \frac{\sqrt{x^2 - 2x} + x}{-2x} = \lim_{x \to \infty} \frac{|x|\sqrt{1 - \frac{2}{x}} + x}{-2x}$$

$$= \lim_{x \to \infty} \frac{x\left[\sqrt{1 - \frac{2}{x}} + 1\right]}{-2x} = \lim_{x \to \infty} \frac{\sqrt{1 - \frac{2}{x}} + 1}{-2} = -1$$

(d) $\lim_{x \to 5} \frac{x}{(5 - x)^3}$

$$\lim_{x \to 5-} \frac{x}{(5-x)^3} = \infty \qquad \lim_{x \to 5+} \frac{x}{(5-x)^3} = -\infty$$

So, the limit does not exist.

2. (a) Is the function

$$f(x) = \begin{cases} 2x-1\ ; & x > 1 \\ 2x-3\ ; & x \leq 1 \end{cases}$$

differentiable at $x = 1$? Explain your answer.

right derivative : $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^+} \left(2+\frac{2}{h}\right) = +\infty$

left derivative : $\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0^-} 2 = 2$

Since they are not equal f is not differentiable at $x=1$

2nd method : f is differentiable $\Rightarrow$ f is continuous

Since f has a discontinuity at $x=1$, it cannot be differentiable there.

(b) Find all points on the curve $y = \frac{2}{x}$ where the tangent line is perpendicular to the line $y = 8x + 3$.

Slope of the curve : $y' = \frac{-2}{x^2}$

Slope of the line $y = 8x+3$: $y' = +8$

We need to solve the equation

$$-\frac{2}{x^2} = -\frac{1}{8}$$ (slope of the normal line)

$\Rightarrow \quad x = \mp 4$

So, $(-4, -\frac{1}{2})$ and $(4, \frac{1}{2})$ are the points where the line, $y = 8x+3$ is perpendicular to the tangent.

3. (a) Find an equation of the tangent to the given curve $\sin(xy^2) = \frac{\sqrt{2}xy}{\pi}$ at the point $(-\pi, \frac{1}{2})$.

$$\sin(xy^2) = \frac{\sqrt{2}}{\pi}xy \Rightarrow \cos(xy^2).(y^2+2xyy') = \frac{\sqrt{2}}{\pi}(y+xy')$$

$$\Rightarrow y'.\left(2xy\cos(xy^2) - \frac{\sqrt{2}}{\pi}x\right) = \frac{\sqrt{2}}{\pi}\cdot y - y^2.\cos(xy^2)$$

$$y'\Big|_{(-\pi,\frac{1}{2})} = \frac{\frac{\sqrt{2}}{\pi}\cdot y - y^2.\cos(xy^2)}{2xy\cos(xy^2)}\Bigg|_{(-\pi,\frac{1}{2})} = \frac{\frac{\sqrt{2}}{\pi}.\left(\frac{1}{2}\right) - \frac{1}{4}.\cos\left(-\frac{\pi}{4}\right)}{2.(-\pi).\frac{1}{2}.\cos\left(-\frac{\pi}{4}\right)} = \frac{-(4-\pi)}{4\pi^2}$$

$\Rightarrow$ tangent $y = \frac{-(4-\pi)}{4\pi^2}(x+\pi) + 1/2$

(b) Show that the equation $x^3 - 4x - 2 = 0$ has a solution in the interval $[-1, 0]$.

The function $f(x) = x^3 - 4x - 2$ is a polynomial and is therefore continuous everywhere.

Now

$f(-1) = -1 + 4 - 2 = 1 > 0$ and

$f(0) = -2 < 0$

Since 0 lies between -2 and 1, the IVP assures us that there must be a number c in $[-1, 0]$ such that

$f(c) = 0$.

(a) Solve the initial value problem

$$\begin{cases} y'(x) = \frac{1}{x^3} - \frac{1}{x^4} \\ y(1) = 1 \end{cases}$$

Where the solution is valid?

$y(x) = \int \left(\frac{1}{x^3} - \frac{1}{x^4}\right) dx = \int (x^{-3} - x^{-4}) dx$

$y(x) = \frac{x^{-2}}{-2} - \frac{x^{-3}}{-3} + C$

$y(x) = \frac{-1}{2x^2} + \frac{1}{3x^3} + C$

$y(1) = 1 = \frac{-1}{2} + \frac{1}{3} + C \quad \Rightarrow C = \frac{7}{6}$

$y(x) = \frac{-1}{2x^2} + \frac{1}{3x^3} + \frac{7}{6}$

The solution is valid on $(0, \infty)$

(b) Show that the function $f(x) = x + 2x^3$ has an inverse and find $(f^{-1})'(3)$.

$f(x) = x + 2x^3 \Rightarrow f'(x) = 1 + 6x^2 > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow f$ is increasing and hence one-to-one

$\Rightarrow f$ is invertible

$y = f^{-1}(x) \Rightarrow x = f(y) = y + 2y^3$

$1 = y' + 6y^2 y'$

$y' = \frac{1}{1+6y^2}$

$x = f(1) = 3 \Rightarrow y = f^{-1}(3) = 1$

$\Rightarrow (f^{-1})'(3) = \frac{1}{1+6y^2}\Big|_{y=1} = \frac{1}{7}$