25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 05.11.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. Evaluate the following limits

a) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x^2 - 5x + 6} = \lim\limits_{x \to 2} \dfrac{\cancel{(x-2)}.(x+2)}{\cancel{(x-2)}.(x-3)} = \lim\limits_{x \to 2} \dfrac{x+2}{x-3} = -4 \,//$

b) $\lim\limits_{x \to 0} \dfrac{|2x - 1| - |2x + 1|}{x} = \lim\limits_{x \to 0} \dfrac{-2x+1-2x-1}{x} = \lim\limits_{x \to 0} \dfrac{-4x}{x} = -4 \,//$

c) $\lim\limits_{x \to \infty} \dfrac{1}{\sqrt{4x^2 - 2x} - 2x} = \lim\limits_{x \to \infty} \dfrac{\sqrt{4x^2-2x}+2x}{4x^2-2x-4x^2}$

$= \lim\limits_{x \to \infty} \dfrac{x.\sqrt{4-\frac{2}{x}}+2x}{-2x}$

$= \lim\limits_{x \to \infty} \dfrac{\cancel{x}.\left[\sqrt{4-\frac{2}{x}}+2\right]}{-2\cancel{x}} = -2 \,//$

d) $\lim\limits_{x \to 3} \dfrac{\sqrt{9 - 6x + x^2}}{x - 3} = \lim\limits_{x \to 3} \dfrac{\sqrt{(x-3)^2}}{x-3} = \lim\limits_{x \to 3} \dfrac{|x-3|}{x-3}$

$\lim\limits_{x \to 3^+} \dfrac{x-3}{x-3} = 1$

Does not exist

$\lim\limits_{x \to 3^-} \dfrac{-(x-3)}{x-3} = -1$

2. (a) Find the constants a and b so that the function

$$f(x) = \begin{cases} ax^3 + b & ; \quad x \le -1 \\ 2x^2 + a & ; \quad -1 < x \le 0 \\ \frac{x}{\sqrt{x+1}-\sqrt{2x+1}} & ; \quad x > 0 \end{cases}$$

is continuous for all x.

If $\lim_{x \to -1} f(x) = f(-1)$ and $\lim_{x \to 0} f(x) = f(0)$ are satisfied then $f(x)$ is continuous for all x. Continuity at $x = -1$:

$\lim_{x \to -1^-} (ax^3 + b) = -a + b$, $\lim_{x \to -1^+} (2x^2 + a) = 2 + a$. So, $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^+} f(x) \Rightarrow -a + b = 2 + a$
$\Rightarrow -2a + b = 2$

Continuity at $x = 0$:

$\lim_{x \to 0^-} (2x^2 + a) = a$, $\lim_{x \to 0^+} \frac{x}{\sqrt{x+1} - \sqrt{2x+1}} = \lim_{x \to 0^+} \frac{x(\sqrt{x+1} + \sqrt{2x+1})}{-x} = \lim_{x \to 0^+} -(\sqrt{x+1} + \sqrt{2x+1}) = -2$

So, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) \Rightarrow \boxed{a = -2}$

If $-2a + b = 2$ and $a = -2$, then $\boxed{b = -2}$.

(b) Use the definition of derivative to find the slope of the curve $y = \sqrt{8 - x^2}$ at $x = 2$.

Let $f(x) = \sqrt{8 - x^2}$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{\sqrt{4 - 4h - h^2} - 2}{h}$$

$$= \lim_{h \to 0} \frac{(\sqrt{4 - 4h - h^2} - 2) \cdot (\sqrt{4 - 4h - h^2} + 2)}{h \cdot (\sqrt{4 - 4h - h^2} + 2)} = \lim_{h \to 0} \frac{-h(4+h)}{h(\sqrt{4 - 4h - h^2} + 2)}$$

$$= \lim_{h \to 0} \frac{-(4+h)}{\sqrt{4 - 4h - h^2} + 2} = -1$$

So, the slope of the given curve at $x = 2$ is -1.

3. (a) Apply the Mean Value Theorem to the function $f(x) = \sin x$ to show that

$$\sin x > x \qquad \text{for } x < 0.$$

If $f(x) = \sin x$, then $f'(x) = \cos x$.
By the MVT there exists a $c \in (x, 0)$ such that:

$$f'(c) = \frac{f(0) - f(x)}{0 - x}.$$

So,

$$\cos c = \frac{0 - \sin x}{0 - x} = \frac{-\sin x}{-x} = \frac{\sin x}{x}.$$

Note that $\cos c < 1$. That implies:

$$1 > \cos c = \frac{\sin x}{x} \Rightarrow 1 > \frac{\sin x}{x}.$$

Multiply both side by $x < 0$,

$$x < \sin x \quad \boxtimes$$

(b) Solve the second order initial value problem

$$\begin{cases} y'' = x + \sin x \\ y(0) = 2 \\ y'(0) = 0 \end{cases}$$

$$y' = \int (x + \sin x)\,dx = \frac{x^2}{2} - \cos x + C_1$$

$$y'(0) = 0 - \cos 0 + C_1 = 0 \Rightarrow \boxed{C_1 = 0}$$

$$y = \int \left(\frac{x^2}{2} - \cos x + 1\right)dx = \frac{x^3}{6} - \sin x + x + C_2$$

$$y(0) = 0 - \sin 0 + 0 + C_2 \Rightarrow \boxed{C_2 = 2}$$

$$y = \frac{x^3}{6} - \sin x + x + 2 \quad \boxtimes$$

4. (a) Find $\frac{d}{dx}\left(\frac{\sqrt{x^2-1}}{x^2+1}\right)\Big|_{x=-2}$

$$= \frac{\frac{2x}{2.\sqrt{x^2-1}}\cdot(x^2+1) - 2x.\sqrt{x^2-1}}{(x^2+1)^2}\Bigg|_{x=-2}$$

$$= \frac{\frac{-2}{\sqrt{3}}\cdot 5 - 2.(-2).\sqrt{3}}{25} = \frac{\frac{-10}{\sqrt{3}}+4\sqrt{3}}{25} = \frac{\frac{2}{\sqrt{3}}}{25} = \frac{2}{25.\sqrt{3}} \quad //$$

(b) Find $\frac{dy}{dx}$ for $x^2 + 4(y-1)^2 = 4$.

$$\Rightarrow 2x + 8.(y-1).y' = 0$$

$$\Rightarrow \frac{dy}{dx} = y' = \frac{-4x}{8.(y-1)} = -\frac{x}{4.(y-1)} = \frac{x}{4.(1-y)} \quad //$$