25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 07.11.2009

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FIRST MIDTERM EXAM

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1. (a) Evaluate the following limit

$$\lim_{x\to-\infty}(x+\sqrt{x^2-6x+1})$$

Solution.

$$\begin{aligned}
\lim_{x\to-\infty}(x+\sqrt{x^2-6x+1}) &= \lim_{x\to-\infty}\frac{(x+\sqrt{x^2-6x+1})\cdot(x-\sqrt{x^2-6x+1})}{(x-\sqrt{x^2-6x+1})}\\
&= \lim_{x\to-\infty}\frac{x^2-x^2+6x-1}{(x-\sqrt{x^2-6x+1})}\\
&= \lim_{x\to-\infty}\frac{6x-1}{(x-\sqrt{x^2-6x+1})}\\
&= \lim_{x\to-\infty}\frac{6x-1}{(x-|x|\sqrt{1-\frac{6}{x}+\frac{1}{x^2}})}\\
&= \lim_{x\to-\infty}\frac{6x-1}{(x+x\cdot\sqrt{1-\frac{6}{x}+\frac{1}{x^2}})}\\
&= \lim_{x\to-\infty}\frac{6-\frac{1}{x}}{(1+\sqrt{1-\frac{6}{x}+\frac{1}{x^2}})}=3.
\end{aligned}$$

(b) Find a and b so that

$$f(x)=\begin{cases}\dfrac{x^2-1}{x+1} & ;\quad x<-1\\ a+2x^2 & ;\quad -1\le x<2\\ bx-1 & ;\quad x\ge 2\end{cases}$$

is continuous for all x.

Solution. If

$$\begin{aligned}&\lim_{x\to-1} f(x)=f(-1)\\ &\lim_{x\to 2} f(x)=f(2)\end{aligned}$$

are satisfied then $f(x)$ is continuous for all x.
Continuity at $x=-1$;

$$\begin{aligned}&\lim_{x\to-1^-} f(x)=\lim_{x\to-1^-}\left(\tfrac{x^2-1}{x+1}\right)=\lim_{x\to-1^-}(x-1)=-2\\ &\lim_{x\to-1^+} f(x)=\lim_{x\to-1^+}(a+2x^2)=a+2\end{aligned}$$

So,

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^-} f(x) \Rightarrow a+2=-2\Rightarrow \quad a=-4$$

Then $\lim_{x\to-1} f(x)=f(-1)$, if $a=-4$.
Continuity at $x=2$;

$$\begin{aligned}&\lim_{x\to2^-} f(x)=\lim_{x\to2^-}(a+2x^2)=a+8\\ &\lim_{x\to2^+} f(x)=\lim_{x\to2^+}(bx-1)=2b-1\end{aligned}$$

So,

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^-} f(x) \Rightarrow a+8=2b-1\Rightarrow \quad b=\tfrac{5}{2}$$

Then $\lim_{x\to2} f(x)=f(2)$, if $b=\frac{5}{2}$.

2. (a) Use the limit definition of derivative to find the equation of the straight line tangent to the curve $y = \sin(3x)$ at $x = 0$.

Solution. Let $f(x) = \sin(3x)$ then by the definition of the derivative

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \tfrac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \tfrac{f(h)-f(0)}{h} \\ &= \lim_{h\to 0} \tfrac{\sin(3h)}{h} \\ &= 3 \cdot \lim_{h\to 0} \tfrac{\sin(3h)}{3h} \\ &= 3 \end{aligned}$$

Since $\lim_{h\to 0} \frac{\sin(3h)}{3h} = 1$. So, the slope of the curve $f(x) = \sin(3x)$ at the point $x = 0$ is $m = 3$, then the equation of the tangent line to the curve $f(x)$ at $(0, 0)$ is

$$\begin{aligned} y &= m(x - x_0) + y_0 \\ y &= 3(x - 0) + 0 \\ y &= 3x \end{aligned}$$

(b) Find the equations of tangent and normal lines to the curve $xy^3 + \tan(x + y) = 1$ at the point $\left(\frac{\pi}{4}, 0\right)$.

Solution. We use implicit differentiation to evaluate the slope of the tangent line.

$$y^3 + 3y^2y'x + (\sec^2(x + y)) \cdot (1 + y') = 0$$

Hence we have

$$y'(3xy^2 + \sec^2(x + y)) = -y^3 - \sec^2(x + y) \implies y' = -\frac{y^3 + \sec^2(x + y)}{3xy^2 + \sec^2(x + y)}.$$

Therefore the slope of the tangent line is

$$m = y'|_{(\frac{\pi}{4},0)} = -\frac{\sec^2 \pi/4}{\sec^2 \pi/4} = -1.$$

So the equation of the tangent line at the point $(\frac{\pi}{4}, 0)$ is $x + y = \frac{\pi}{4}$.

The slope of the normal line is 1. The equation of the normal line at the point $(\frac{\pi}{4}, 0)$ is given by $x - y = \frac{\pi}{4}$.

3. (a) Use the Mean Value Theorem to show that $\cos x > 1 - \dfrac{x^2}{2}$ for $x < 0$.

Solution. Let $f(x) = \cos(x) + \frac{x^2}{2}$. Then $f(x)$ is continuous and differentiable for all x. Applying Mean Value Theorem to the function $f(x) = \cos(x) + \frac{x^2}{2}$ on the interval $[x, 0]$ yields that there exist a constant $c \in (x, 0)$ such that

$$\begin{aligned} \frac{f(x) - f(0)}{x - 0} &= f'(c) \\ \frac{\cos(x) + \frac{x^2}{2} - 1}{x} &= -\sin(c) + c \end{aligned}$$

Since $\sin(x) > x$ for all $x < 0$, then $-\sin(c) + c < 0$ for $c \in (x, 0)$ and hence

$$\frac{\cos(x) + \frac{x^2}{2} - 1}{x} = -\sin(c) + c < 0$$

this implies $\cos(x) + \dfrac{x^2}{2} - 1 > 0$, since $x < 0$. Consequently, $\cos x > 1 - \dfrac{x^2}{2}$.

(b) Use the Intermediate Value Theorem to show that $3x^3 + 2x - 1 = 0$ has one solution in $[0, 2]$.

Solution. Let $f(x) = 3x^3 + 2x - 1$. Since $f(x)$ is a polynomial function then it is continuous everywhere, so it is continuous on the given interval $[0, 2]$.

Check

$$\begin{aligned} f(0) &= -1 < 0 \\ f(2) &= 27 > 0 \end{aligned}$$

Since $f(0) < 0 < f(2)$, by Intermediate Value Theorem there exist $s \in [0, 2]$ such that

$$f(s) = 0$$

That means $f(x)$ has one solution in [0,2].

4. (a) Solve the initial value problem

$$\begin{cases} y' = \dfrac{-1}{x^2} + \dfrac{1}{3}x^{-2/3} \\ y(-1) = 0 \end{cases}$$

Where the solution is valid?

Solution. To find the solution, we must take the antiderivative of y'

$$\begin{array}{rcl} y & = & \int (\frac{-1}{x^2} + \frac{1}{3}x^{-2/3})dx \\ y & = & \frac{1}{x} + x^{1/3} + c \end{array}$$

for any constant c. Substituting into $y(-1) = 0$; $y(-1) = -2 + c = 0$, So $c = 2$. Then the solution is

$$y = \frac{1}{x} + x^{1/3} + 2$$

which is valid on $(-\infty, 0)$.

(b) Calculate enough derivatives of the function $f(x) = \dfrac{1}{2-x}$ to enable you to guess the general formula for the nth derivative of f. Then verify your guess using mathematical induction.

Solution.

$$\begin{array}{rcl} f(x) & = & \frac{1}{2-x} \\ f'(x) & = & \frac{1}{(2-x)^2} \\ f''(x) & = & \frac{2!}{(2-x)^3} \\ f'''(x) & = & \frac{3!}{(2-x)^4} \\ & \vdots & \\ f^{(n)}(x) & = & \frac{n!}{(2-x)^{(n+1)}} \qquad (*) \end{array}$$

To prove it, we use the mathematical induction.

For $n = 1$, $f'(x) = \frac{1}{(2-x)^2}$ is satisfied.

Assume that $f^{(n)}(x) = \frac{n!}{(2-x)^{(n+1)}}$ is true for $n = k$, then by differentiating $f^{(k)}(x)$,

$$f^{(k+1)}(x) = (f^{(k)})'(x) = \left(\frac{k!}{(2-x)^{(k+1)}}\right)' = \frac{(k+1)k!}{(2-x)^{(k+2)}} = \frac{(k+1)!}{(2-x)^{(k+2)}}$$

So we have found that

$$f^{(k+1)}(x) = \frac{(k+1)!}{(2-x)^{(k+2)}}$$

which is equivalent to $f^{(k+1)}(x)$ by substituting $k+1$ instead of n in $(*)$. Hence, we proved

$$f^{(n)}(x) = \frac{n!}{(2-x)^{(n+1)}}.$$