25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 12.01.2012

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

KEY

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1. (a) Use a suitable linearization to approximate $\frac{1}{(83)^{\frac{1}{4}}}$

Let $f(x) = \frac{1}{x^{1/4}} = x^{-1/4}$. Then $f'(x) = -\frac{1}{4} x^{-5/4}$ and $f(81) = \frac{1}{3}$,

$f'(81) = -\frac{1}{4.3^5}$

$L(x) = f(a) + f'(a).(x-a)$

$= f(81) + f'(81)(x-81) = \frac{1}{3} - \frac{1}{4.3^5}(x-81)$

$\Rightarrow L(83) = \frac{1}{3} - \frac{1}{4.3^5}(83-81) = \frac{1}{3} - \frac{1}{4.3^5}.2 = \frac{161}{486}$

(b) Find the 7^{th} order Taylor polynomial of $f(x) = \sin(2x)$ about $x = \frac{\pi}{2}$

$f(x) = \sin 2x$, $f'(x) = 2\cos 2x$, $f''(x) = -2^2 \sin 2x$, $f'''(x) = -2^3 \cos 2x$,

$f^{iv}(x) = 2^4 \sin 2x$, $f^{(5)}(x) = 2^5 \cos 2x$, $f^{(6)}(x) = -2^6 \sin 2x$, $f^{(7)}(x) = -2^7 \cos 2x$

$f(\frac{\pi}{2}) = f''(\frac{\pi}{2}) = f^{iv}(\frac{\pi}{2}) = f^{vi}(\frac{\pi}{2}) = 0$ $\quad f'(\frac{\pi}{2}) = -2$, $f'''(\frac{\pi}{2}) = 8$

$f^{v}(\frac{\pi}{2}) = -32$, $f^{(7)}(\frac{\pi}{2}) = 128$

$P_7(\frac{\pi}{2}) = f(\frac{\pi}{2}) + f'(\frac{\pi}{2}).\frac{x-\frac{\pi}{2}}{1!} + f''(\frac{\pi}{2}).\frac{(x-\frac{\pi}{2})^2}{2!} + \dots + f^{(7)}(\frac{\pi}{2}).\frac{(x-\frac{\pi}{2})^7}{7!}$

$= -2.(x-\frac{\pi}{2}) + 8.\frac{(x-\frac{\pi}{2})^3}{3!} - 32.\frac{(x-\frac{\pi}{2})^5}{5!} + 128.\frac{(x-\frac{\pi}{2})^7}{7!}$

2. Let P_n denote the partition of the interval $[0,1]$ into n subintervals of equal length $\Delta x = \frac{1}{n}$. For a given function $f(x) = 2x+1$

(a) Calculate the Lower Riemann sum $L(f, P_n)$.

$$L(f,P_n) = \sum_{i=0}^{n-1} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=0}^{n-1} \frac{1}{n}\left[2\cdot\frac{i}{n}+1\right] = \frac{2}{n^2}\sum_{i=0}^{n-1} i + \frac{1}{n}\sum_{i=0}^{n-1} 1$$

$$= \frac{2}{n^2}\cdot\frac{(n-1)\cdot n}{2} + \frac{1}{n}\cdot n = \frac{2n-1}{n}$$

(b) Calculate the Upper Riemann sum $U(f, P_n)$.

$$U(f,P_n) = \sum_{i=1}^{n} \frac{1}{n} f\left(\frac{i}{n}\right) = \sum_{i=1}^{n} \frac{1}{n}\cdot\left[2\cdot\frac{i}{n}+1\right] = \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} 1$$

$$= \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n}\cdot n = \frac{2n+1}{n}$$

(c) Show that $\lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$. Use the obtained information to calculate $\int_0^1 (2x+1)dx$.

$$\left.\begin{aligned} \lim_{n\to\infty} L(f,P_n) &= \lim_{n\to\infty} \frac{2n-1}{n} = 2 \\ \lim_{n\to\infty} U(f,P_n) &= \lim_{n\to\infty} \frac{2n+1}{n} = 2 \end{aligned}\right\} \Rightarrow \int_0^1 (2x+1)\,dx = 2$$

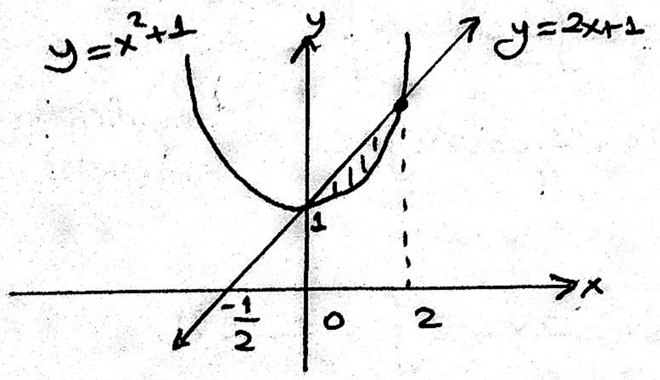
3. (a) Find $H'(9)$ if $H(x) = 2x \int_3^{\sqrt{x}} e^{-t^2}\, dt$

$$H'(x) = 2\int_3^{\sqrt{x}} e^{-t^2}\, dt + 2x \cdot e^{-x} \cdot \frac{1}{2\sqrt{x}}$$

$$= 2\int_3^{\sqrt{x}} e^{-t^2}\, dt + \sqrt{x} \cdot e^{-x}$$

$$\Rightarrow H'(9) = 2\int_3^{3} e^{-t^2}\, dt + 3e^{-9} = 2.0 + 3e^{-9} = 3e^{-9}$$

(b) Sketch and find the area of the plane region bounded by the curves $f(x) = x^2+1$ and $g(x) = 2x + 1$.



$$\left.\begin{aligned} y &= x^2+1 \\ y &= 2x+1 \end{aligned}\right\} \Rightarrow x^2+1 = 2x+1$$

$$x^2 - 2x = 0$$

$$x = 0 \text{ or } x = 2$$

$$A = \int_0^2 \left[(2x+1) - (x^2+1) \right] dx = \int_0^2 \left[2x - x^2 \right] dx$$

$$= \left(x^2 - \frac{x^3}{3} \right) \Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

4. Evaluate the given integrals

a) $\int \frac{x\ln(x^2+1)}{(x^2+1)}dx$ Let $u = \ln(x^2+1)$. Then $\frac{du}{2} = \frac{x}{x^2+1}dx$

$$\int \frac{x\cdot\ln(x^2+1)}{x^2+1}dx = \int u\,\frac{du}{2} = \frac{1}{4}u^2 + c = \frac{1}{4}\left[\ln(x^2+1)\right]^2 + C$$

b) $\int x\cos^2 x\,dx$

$$\int x\cos^2 x\,dx = \int x\left(\frac{1+\cos 2x}{2}\right)dx = \frac{1}{2}\int x\,dx + \frac{1}{2}\int x\cos 2x\,dx$$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{1}{2}x\sin 2x - \int \frac{1}{2}\sin 2x\,dx\right]$$

$x = u$ $\quad \cos 2x\,dx = dv$

$dx = du$ $\quad \frac{1}{2}\sin 2x = v$

$$= \frac{x^2}{4} + \frac{1}{2}\left[\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right] + C$$

(integration by parts)

c) $\int \frac{2x^2}{x^2+x-2}dx$

$$\int \frac{2x^2}{x^2+x-2}dx = 2\int\left(1 - \frac{x-2}{x^2+x-2}\right)dx = 2x - 2\underbrace{\int \frac{x-2}{x^2+x-2}dx}_{I}$$

If $\frac{x-2}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1} = \frac{A(x-1)+B(x+2)}{x^2+x-2}$ then $A+B=1$ and $-A+2B=-2$,

so that $A = \frac{4}{3}$ and $B = \frac{-1}{3}$. Thus $I = \int \frac{\frac{4}{3}}{x+2}dx + \int \frac{-\frac{1}{3}}{x-1}dx$

$$= \frac{4}{3}\ln|x+2| - \frac{1}{3}\ln|x-1| + C$$

d) $\int \frac{\sqrt{4+x^2}}{x^4}dx$

$x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

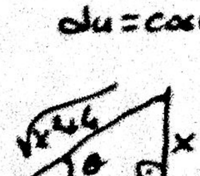
$$\int \frac{\sqrt{4+x^2}}{x^4}dx = \int \frac{(2\sec\theta)(2\sec^2\theta)d\theta}{16\tan^4\theta} = \frac{1}{4}\int\frac{\sec^3\theta}{\tan^4\theta}d\theta = \frac{1}{4}\int\frac{\cos\theta}{\sin^4\theta}d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$$= \frac{1}{4}\int\frac{du}{u^4} = \frac{1}{4}\int u^{-4}\,du = \frac{-1}{12u^3} + C$$

$$= -\frac{1}{12(\sin\theta)^3} + C = -\frac{\left(\sqrt{x^2+4}\right)^{3/2}}{12x^3} + C$$



$\sin\theta = \frac{x}{\sqrt{x^2+4}}$

$$= -\frac{\sqrt{(x^2+4)^3}}{12x^3} + C$$