25 points	25 points	25 points	25 points	100 points
1	2	3	4	**Total**

MATH 153 CALCULUS I 15.01.2010

İzmir University of Economics Faculty of Arts and Science Department of Mathematics

FINAL EXAM

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Duration: 110 mins

1. (a) Evaluate $\int \frac{\cos x}{(\sin x)^{1/2}(1+(\sin x)^{1/3})}dx$

Solution :

Let $\sin x = u \quad \Rightarrow \cos x dx = du$ then

$$\int \frac{\cos x}{(\sin x)^{1/2}(1+(\sin x)^{1/3})}dx = \int \frac{du}{u^{1/2}(1+u^{1/3})}dx$$

Let $u = s^6 \quad \Rightarrow du = 6s^5 ds$, then

$$\begin{aligned}
\int \frac{du}{u^{1/2}(1+u^{1/3})}dx &= 6\int \frac{s^5 ds}{s^3(1+s^2)} = 6\int \frac{s^2 ds}{1+s^2} \\
&= 6\int \left(1 - \frac{1}{1+s^2}\right) ds = 6(s - \tan^{-1} s) + C \\
&= 6(u^{\frac{1}{6}} - \tan^{-1} u^{\frac{1}{6}}) + C = 6((\sin x)^{\frac{1}{6}} - \tan^{-1}(\sin x)^{\frac{1}{6}}) + C.
\end{aligned}$$

(b) Evaluate $\int_1^{\sqrt{e}} \sin(\ln(x^2))dx$

Solution : Let $\ln(x^2) = 2\ln(x) = s \quad \Rightarrow \frac{s}{2} = \ln(x)$

and $x = e^{\frac{s}{2}} \quad \Rightarrow ds = \frac{2dx}{x} \quad \Rightarrow dx = \frac{1}{2}e^{\frac{s}{2}}ds$, then

$$\int \sin(\ln(x^2))dx = \frac{1}{2}\int e^{\frac{s}{2}} \sin s ds$$

Let $I = \int e^{\frac{s}{2}} \sin s ds$

$$\begin{matrix} u = \sin s, & du = \cos s ds \\ dv = e^{\frac{s}{2}} ds, & v = 2e^{\frac{s}{2}} \end{matrix} \quad \Rightarrow I = 2e^{\frac{s}{2}} \sin s - 2\int e^{\frac{s}{2}} \cos s ds$$

Let $I_1 = \int e^{\frac{s}{2}} \cos s ds$

$$\begin{matrix} u = \cos s, & du = -\sin s ds \\ dv = e^{\frac{s}{2}} du, & v = 2e^{\frac{s}{2}} \end{matrix} \quad \Rightarrow I_1 = 2e^{\frac{s}{2}} \cos s + 2\int e^{\frac{s}{2}} \sin s ds$$

$$\begin{aligned}
\text{Hence,} \quad I &= 2e^{\frac{s}{2}} \sin s - 2\left(2e^{\frac{s}{2}} \cos s + 2\int e^{\frac{s}{2}} \sin s ds\right) \\
&= 2e^{\frac{s}{2}} \sin s - 4e^{\frac{s}{2}} \cos s - 4\int e^{\frac{s}{2}} \sin s ds
\end{aligned}$$

$$\begin{aligned}
5I &= 2e^{\frac{s}{2}} \sin s - 4e^{\frac{s}{2}} \cos s \\
I &= \frac{2}{5}e^{\frac{s}{2}} \sin s - \frac{4}{5}e^{\frac{s}{2}} \cos s
\end{aligned}$$

Hence, $\frac{1}{2}\int e^{\frac{s}{2}} \sin s ds = \frac{1}{5}e^{\frac{s}{2}} \sin s - \frac{2}{5}e^{\frac{s}{2}} \cos s + C$, where C is any arbitrary constant. So,

$$\begin{aligned}
\int_1^{\sqrt{e}} \sin(\ln(x^2))dx &= \left(\frac{1}{5}x \sin(\ln(x^2)) - \frac{2}{5}x \cos(\ln(x^2))\right)\Big|_1^{\sqrt{e}} \\
&= \left(\frac{1}{5}\sqrt{e}\sin 1 - \frac{2}{5}(\sqrt{e}\cos 1 - 1)\right).
\end{aligned}$$

2. (a) Evaluate $\int e^x(1-e^{2x})^{1/2}dx$

Solution : Let $e^x = u \quad \Rightarrow e^x dx = du$, then

$$\int e^x(1-e^{2x})^{1/2}dx = \int (1-u^2)^{1/2}du$$

Let $u = \sin\theta \quad \Rightarrow d\theta = \cos\theta d\theta$, then

$$\int (1-u^2)^{1/2}du = \int (1-(\sin\theta)^2)^{1/2}\cos\theta d\theta = \int \cos^2\theta d\theta$$

Since, $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$ then

$$\int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2}d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 2\theta d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C$$

Since, $u = \sin\theta \quad \Rightarrow \cos\theta = \sqrt{1-u^2}$.
So, $\sin 2\theta = 2\sin\theta\cos\theta = 2u\sqrt{1-u^2}$ And also, $\theta = \sin^{-1} u$.
Hence,

$$\int (1-u^2)^{1/2}du = \frac{1}{2}\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2} + C$$

and since $e^x = u$

$$\int e^x(1-e^{2x})^{1/2}dx = \frac{1}{2}\sin^{-1} e^x + \frac{1}{2}e^x\sqrt{1-e^{2x}} + C.$$

(b) Evaluate $\int \dfrac{dx}{x^4-1}$

Solution : The partial fraction decomposition is

$$\begin{aligned}\frac{1}{x^4-1} &= \frac{A}{x-1} + \frac{B}{x+1} + \frac{Cx+D}{x^2+1}\\ &= \frac{A(x^3+x^2+x+1) + B(x^3-x^2+x-1) + C(x^3-x) + D(x^2-1)}{x^4-1}\end{aligned}$$

$$\Rightarrow \begin{cases} A+B+C=0\\ A-B+D=0\\ A+B-C=0\\ A-B-D=1\end{cases}, \quad \text{So } A = \frac{1}{4},\ B = -\frac{1}{4},\ C=0,\ D=-\frac{1}{2}.$$

Hence,

$$\begin{aligned}\int \frac{dx}{x^4-1} &= \frac{1}{4}\int\frac{dx}{x-1} - \frac{1}{4}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{x^2+1}\\ &= \frac{1}{4}\ln|x-1| - \frac{1}{4}\ln|x+1| - \frac{1}{2}\tan^{-1}x + C_1\\ &= \frac{1}{4}\ln\frac{|x-1|}{|x+1|} - \frac{1}{2}\tan^{-1}x + C_1\end{aligned}$$

where, C_1 is any arbitrary constant.

3. (a) Find the area of the plane region bounded above by $2y = 4x - x^2$ and below by $2y + 3x = 6$.

Solution :

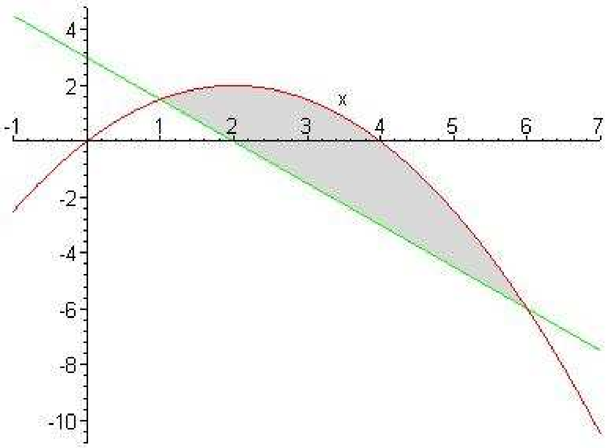
Firstly, we solve the equation $4x - x^2 = 6 - 3x$ to find the limits of the integration

$$4x - x^2 = 6 - 3x \quad \Rightarrow x^2 - 7x + 6 = 0 \quad \Rightarrow x = 1 \text{ and } x = 6.$$

Then

$$\begin{aligned}
\text{Area} = \int_1^6 \left|\frac{1}{2}\left(4x - x^2\right) - \frac{1}{2}\left(6 - 3x\right)\right| dx &= \int_1^6 \left|\left(2x - \frac{x^2}{2}\right) - \left(3 - \frac{3}{2}x\right)\right| dx \\
&= \int_1^6 \left(\frac{7}{2}x - \frac{x^2}{2} - 3\right) dx \\
&= \left(\frac{7}{4}x^2 - \frac{x^3}{6} - 3x\right)\Big|_1^6 \\
&= \frac{7}{4}(36 - 1) - \frac{1}{6}(216 - 1) - 3(6 - 1) \\
&= \frac{125}{12} \text{ sq. units}
\end{aligned}$$

The graph of the plane region is as the following



(b) Find $g'(4)$ if $g(x) = \sqrt{x} \displaystyle\int_2^{\sqrt{x}} e^{t^2} dt$

Solution : Differentiating the function $g(x)$ yields

$$g'(x) = \frac{1}{2\sqrt{x}}\left(\int_2^{\sqrt{x}} e^{t^2} dt\right) + \sqrt{x}\left[\left(e^{(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}}\right)\right]$$

$$\begin{aligned}
\text{Hence, } g'(4) &= \frac{1}{2\sqrt{4}}\left(\int_2^{\sqrt{4}} e^{t^2} dt\right) + \sqrt{4}\left[\left(e^{(\sqrt{4})^2} \cdot \frac{1}{2\sqrt{4}}\right)\right] \\
&= \frac{1}{4}\left(\int_2^{2} e^{t^2} dt\right) + 2\left[\left(e^4 \cdot \frac{1}{4}\right)\right] \\
&= \left(\frac{1}{4} \cdot 0\right) + \frac{1}{2}e^4 = \frac{1}{2}e^4
\end{aligned}$$

4. (a) Express the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos\left(\frac{\pi i}{n}\right)$$

as a definite integral and evaluate the obtained integral.

Solution :

Since

$$\lim_{n\to\infty}\sum_{i=1}^{n}\Delta x_i f(x_i)=\int_a^b f(x)dx,$$

Here, $\Delta x=\frac{\pi-0}{n}=\frac{\pi}{n}$, $f(x_i)=f(\frac{i\pi}{n})=\cos\left(\frac{i\pi}{n}\right)$.

Hence, $f(x)=\cos x$ and

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{\pi}{n}\cos\left(\frac{\pi i}{n}\right)=\int_0^{\pi}\cos x dx=\sin x\Big|_0^{\pi}=\sin\pi-\sin 0=0.$$

(b) Use a suitable linearization to approximate the value of $\frac{1}{\sqrt{4.16}}$.

Solution :

Let $L(x)$ be a linearization of a function about a then

$$L(x)=f(a)+f'(a)(x-a)$$

Here, $f(x)=\frac{1}{\sqrt{x}}$, $f(4)=\frac{1}{\sqrt{4}}=\frac{1}{2}$;

$f'(x)=-\frac{1}{2}x^{-\frac{3}{2}}, f(4)=-\frac{1}{2}(4)^{-\frac{3}{2}}=-\frac{1}{16}.$

So, the linearization of $f(x)$ about $x=4$ is

$$L(x)=\frac{1}{2}-\frac{1}{16}(x-4)$$

and to find the approximate value of $\frac{1}{\sqrt{4,16}}$, we substitute $4,16$ instead of x :

$$\frac{1}{\sqrt{4,16}}=f(4,16)\approx L(4,16)=\frac{1}{2}-\frac{1}{16}[(4,16)-4]=\frac{1}{2}-\frac{1}{100}=0,49.$$

Hence,

$$\frac{1}{\sqrt{4,16}}\approx 0,49.$$