KEY

| 25 points | 25 points | 25 points | 25 points | 100 points |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | **Total** |
| | | | | |

# MATH 153 CALCULUS I 12.12.2014

İzmir University of Economics Faculty of Arts and Sciences, Department of Mathematics

## Midterm Exam 2

**Student Name and Surname:**...............................

**Section:** **Check for your instructor below:**

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(1) Evaluate the limits:

(a) $\lim_{x\to 0} \dfrac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$ $\left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{\frac{1}{2}(1+x)^{-\frac{1}{2}}-\frac{1}{2}}{2x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{-\frac{1}{4}(1+x)^{-\frac{3}{2}}}{2} = -\frac{1}{8}$$

(b) $\lim_{x\to 0+} \left(\dfrac{1}{x}-\dfrac{1}{xe^{ax}}\right)$ $(\infty-\infty)$

$$\lim_{x\to 0^+} \frac{e^{ax}-1}{xe^{ax}} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0^+} \frac{ae^{ax}}{e^{ax}+xae^{ax}} = \lim_{x\to 0^+} \frac{a\cancel{e^{ax}}}{\cancel{e^{ax}}(1+\cancel{xa})} = a.$$

(c) $\lim_{x\to 0} (\cos 2x)^{\frac{1}{x}}$

$$y = (\cos 2x)^{\frac{1}{x}}$$

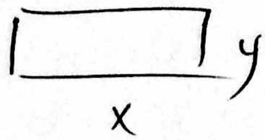
$$\ln y = \frac{1}{x}\cdot \ln(\cos 2x)$$

$$\ln y = \frac{\ln(\cos 2x)}{x} \qquad \lim_{x\to 0} \ln y = \lim_{x\to 0} \frac{\ln(\cos(2x))}{x}$$

$$= \lim_{x\to 0} \frac{\frac{-1}{\cos 2x}\cdot \sin 2x \cdot 2}{1} = \lim_{x\to 0} \frac{-2\tan(2x)}{1} = 0$$

$$y = e^0 = 1$$

(2) (*a*) How fast is the area of a rectangle changing if one side is 15 cm long and is decreasing at a rate of 3 cm/s and the other side is 10 cm long and increasing 4 cm/s?




$$A = x \cdot y.$$

$$x = 15. \qquad y = 10$$

$$\frac{dx}{dt} = {}^{-}3. \qquad \frac{dy}{dt} = 4$$

$$\frac{dA}{dt} = \frac{dx}{dt} \cdot y + \frac{dy}{dt} \cdot x.$$

$$\frac{dA}{dt} = (-3).(10) + (4).(15) = \underline{\underline{30}}$$

(*b*) Two numbers have sum 20. What are the numbers if the product of the square of one and the fourth power of the other is as large as possible?

$$x + y = 20 \implies y = 20 - x.$$

$$A = x^2 \cdot y^4 = x^2 (20 - x)^4$$

$$A' = 2x.(20-x)^4 + 4(20-x)^3.(-1).x^2$$

$$2x(20-x)^3 \left[ (20-x) - 2x \right] = 0$$

$$2x(20-x)^3 . (20-3x) = 0.$$

$$x = 0 \quad x = 20 \quad 3x = 20$$

$$x = \frac{20}{3} \qquad y = 20 - \frac{20}{3} = \frac{60-20}{3} = \frac{40}{3}$$

The numbers are $\frac{20}{3}$ and $\frac{40}{3}$.

(3) Evaluate the integrals:

(a) $\int \frac{dx}{x(1+\ln x)^2}$

$u = 1+\ln x.$

$du = \frac{1}{x} \cdot dx$

$\int \frac{du}{u^2} = \int u^{-2} \cdot du = \frac{u^{-1}}{-1} + C = \frac{-1}{u} + C$

$= \frac{-1}{1+\ln x} + C$

(b) $\int \sqrt{x} \sin^2(x^{\frac{3}{2}} - 1)dx$

$u = x^{\frac{3}{2}} - 1$

$du = \frac{3}{2} x^{1/2} \cdot dx.$

$\frac{2}{3} du = \sqrt{x}\, dx.$

$= \frac{2}{3} \int \sin^2 u \cdot du = \frac{\cancel{2}}{3} \cdot \frac{1}{\cancel{2}} \int (1 - \cos 2u) du$

$= \frac{1}{3}\left(u - \frac{\sin 2u}{2}\right) + C$

$\frac{1}{3}(x^{\frac{3}{2}} - 1) - \frac{1}{6} \sin(2x^{\frac{3}{2}} - 2) + C$

(c) $\int_{\frac{\pi^2}{36}}^{\frac{\pi^2}{4}} \frac{\cos\sqrt{x}dx}{\sqrt{x}\sin\sqrt{x}}$

$= \int \frac{\cos\sqrt{x}\, dx}{\sqrt{x} \cdot (\sin\sqrt{x})^{1/2}} = 2\int_{u=1/2}^{u=1} \frac{du}{u^{1/2}} = 2\int u^{-\frac{1}{2}} du$

$u = \sin\sqrt{x}.$

$du = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} \cdot dx.$

$2du = \frac{\cos\sqrt{x}\, dx}{\sqrt{x}}.$

$x = \frac{\pi^2}{36} \Rightarrow u = \sin\frac{\pi}{6} = \frac{1}{2}$

$x = \frac{\pi^2}{4} \Rightarrow u = \sin\frac{\pi}{2} = 1.$

$4u^{\frac{1}{2}} \Big|_{u=\frac{1}{2}}^{u=1} = 4\left(1 - \frac{1}{\sqrt{2}}\right)$

$= 4 - \frac{4}{\sqrt{2}}$ //

(4) Let $y = \dfrac{x^2}{x^2 - 1}$

(*a*) Determine the domain and intercept(s).

Domain : $\mathbb{R} - \{\mp 1\}$

$x = 0 \Rightarrow y = 0$ (y-int) $(0,0)$

$y = 0 \Rightarrow x^2 = 0 \Rightarrow x = 0$ (x-int.)

(*b*) Determine the asymptotes(if any).

$x = \mp 1$ vertical asym.

$y = 1$ horizontal

$$\lim_{x \to \mp\infty} \frac{x^2}{x^2 - 1} = 1.$$

(*c*) Find the intervals in which the function $f(x)$ is increasing and decreasing.

$$f'(x) = \frac{2x.(x^2-1) - 2x.x^2}{(x^2-1)^2} = \frac{\cancel{2x^3} - 2x - \cancel{2x^3}}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2} = 0.$$

$x = 0$ C.P.

$x = \mp 1$ S.P

| | $-\infty$ | $-1$ | $0$ | $1$ | $\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+$ | $+$ | loc.max | $-$ | $-$ |
| $f(x)$ | $\nearrow$ | $\nearrow$ | | $\searrow$ | $\searrow$ |

at $x = 0$ local max. $(0,0)$.

$(-\infty,-1) \cup (-1, 0)$ : increasing interval.

$(0, 1) \cup (1, \infty)$ ; decreasing interval.

(*d*) Find the intervals in which the function $f(x)$ is concave up and concave down.

$$f''(x) = \frac{-2(x^2-1)^2 - 2(x^2-1)\cdot 2x\cdot(-2x)}{(x^2-1)^4}.$$

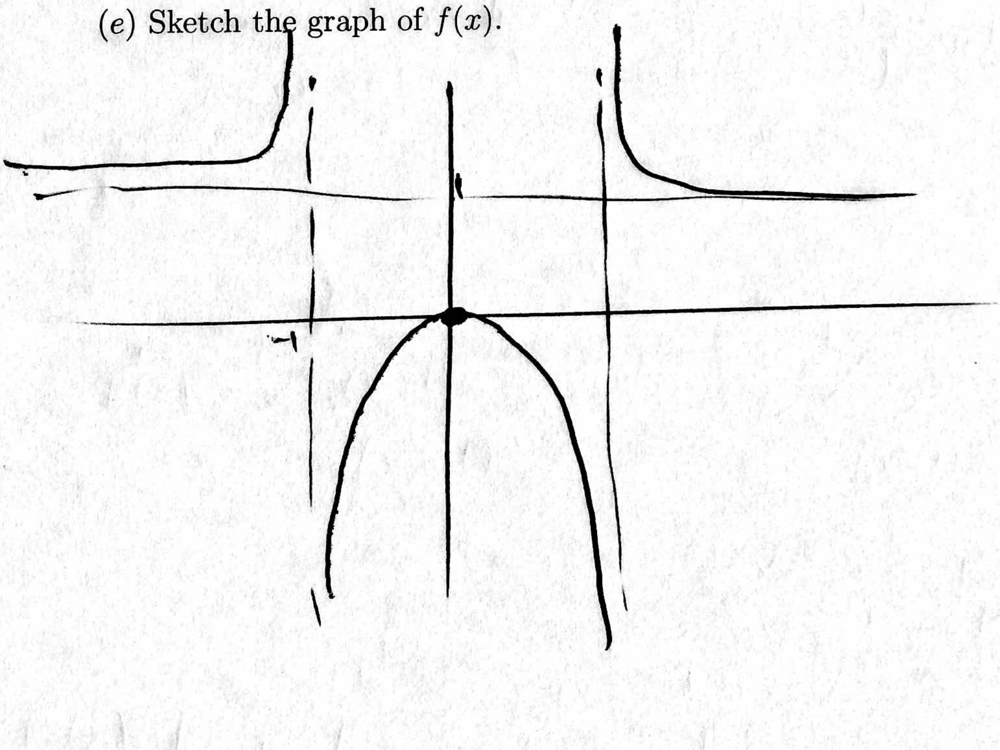
$$\frac{-2\cancel{(x^2-1)}[x^2-1-4x^2]}{(x^2-1)^{\cancel{4}3}} = \frac{+2(1+3x^2)}{(x^2-1)^3}.$$

| | $-\infty$ | | $-1$ | | $1$ | | $\infty$ |
|---|---|---|---|---|---|---|---|
| $f''(x)$ | | $+$ | | $-$ | | $+$ | |
| $f(x)$ | | $\cup$ | | $\cap$ | | $\cup$ | |
| | | CU | | CD | | CU. | |

$(-\infty,-1)\cup(1,\infty)$ Concave Up.

$(-1,1)$ Concave down

(*e*) Sketch the graph of $f(x)$.




$x=-2 \Rightarrow y=\frac{4}{3}$

$x=2 \Rightarrow y=\frac{4}{3}$