IUE – MATH 103 – Fundamentals of Mathematics

2^{nd} Midterm — December 10, 2014 — 11:00 - 12:50

Name: ______________________ **ID #:** ______________________

Q1	Q2	Q3	Q4	TOTAL
25	25	25	25	100

Important: Show all your work. Answers without sufficient explanation might not get full credit. Be neat.

Signature: ______________________

GOOD LUCK!

1. Briefly explain the following terms about a relation on a set:
a) reflexive,
b) symmetric,
c) antisymmetric,
d) transitive,
e) total.

Solution: Let A be a set and $\Re$ be a relation on A.
a) $\Re$ is reflexive if $a\Re a$ for all $a \in A$.
b) $\Re$ is symmetric provided that for all $a, b \in A$, if $a\Re b$, then $b\Re a$.
c) $\Re$ is antisymmetric provided that for all $a, b \in A$, if $a\Re b$ and $b\Re a$, then $a = b$.
d) $\Re$ is transitive provided that for all $a, b, c \in A$, if $a\Re b$ and $b\Re c$, then $a\Re c$.
e) $\Re$ is total if for all $a, b \in A$, either $a\Re b$ or $b\Re a$.

2. Prove by PMI (principle of mathematical induction) that for any $n \geq 2$

$$(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{n}) = \frac{1}{n}.$$

Solution:

Initial step: For $n = 2$, $(1 - 1/2) = 1/2$, so the equation holds for $n = 2$.
Induction step: Suppose that

$$(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k}) = \frac{1}{k}.$$

We will show that the equation holds for $k+1$. By the induction assumption,

$$(1-\frac{1}{2})(1-\frac{1}{3})\cdots(1-\frac{1}{k})(1-\frac{1}{k+1})$$

is equal to

$$\frac{1}{k}(1-\frac{1}{k+1}) = \frac{1}{k}(\frac{k}{k+1}) = \frac{1}{k+1}.$$

Hence, the equation satisfies for all $n \geq 2$.

3. List or describe all equivalence relations that can be defined on $A = \{w, x, y, z\}$.

Solution:

List Method: Any equivalence relation on A is reflexive. Thus, any list of equivalence relations on A must contain $(w, w), (x, x), (y, y), (z, z)$. Note that the relation $R_0 = \{(w, w), (x, x), (y, y), (z, z)\}$ is also symmetric and transitive. So, it is an equivalence relation which consists of elements in the smallest number. Then the other equivalence relations:

$$\begin{aligned}
R_1 &= R_0 \cup \{(x, y), (y, x)\}, \\
R_2 &= R_0 \cup \{(x, z), (z, x)\}, \\
R_3 &= R_0 \cup \{(x, w), (w, x)\}, \\
R_4 &= R_0 \cup \{(y, z), (z, y)\}, \\
R_5 &= R_0 \cup \{(y, w), (w, y)\}, \\
R_6 &= R_0 \cup \{(z, w), (w, z)\}, \\
R_7 &= R_1 \cup R_6, \\
R_8 &= R_2 \cup R_5, \\
R_9 &= R_3 \cup R_4, \\
R_{10} &= R_1 \cup R_4 \cup R_2, \\
R_{11} &= R_1 \cup R_5 \cup R_3, \\
R_{12} &= R_2 \cup R_6 \cup R_3, \\
R_{13} &= R_4 \cup R_5 \cup R_6, \\
R_{14} &= R_7 \cup R_8 \cup R_9,
\end{aligned}$$

Second Method: Let $\mathcal{F} = \{B\}$ be a partition on A. For $a, b \in A$, define

$$a\Re b \text{ if there exists } B \in \mathcal{F} \text{ such that } a, b \in B.$$

This is clearly reflexive and symmetric. Moreover, it is transitive because if $a\Re b$ and $b\Re c$, then there exist $B, B' \in \mathcal{F}$ such that $a, b \in B$ and $b, c \in B'$. But $\mathcal{F}$ is a partition of A. So, $B = B'$. Thus, $\Re$ is an equivalence relation on A. Since any equivalence relation corresponds to such a relation, it describes all equivalence relations on A.

4. Prove or disprove:

$$A - (B \cup C) = (A - B) \cap (A - C)$$

Solution:

$$\begin{aligned} A - (B \cup C) &= A \cap (B \cup C)^c \\ &= A \cap (B^c \cap C^c) \\ &= (A \cap B^c) \cap (A \cap C^c) \\ &= (A - B) \cap (A - C). \end{aligned}$$