MATH 103 – Homework Assignment III

1. Suppose $f : A \to B$ is a function, and let A_1 and A_2 be subsets of A. In the following two set equality statements, three of the four subset inclusion statements are true, and one is false. Prove the three that are true, and provide a counterexample to demonstrate that the fourth is false.

a) $f[A_1 \cap A_2] = f[A_1] \cap f[A_2]$.

b) $f[A_1 \cup A_2] = f[A_1] \cup f[A_2]$.

2. Suppose $f : A \to B$ is a function, $A_1 \subset A$ and $B_1 \subset B$. Prove the following.

a) $A_1 \subset f^{-1}[f[A_1]]$.

b) $f^{-1}[f[B_1]] \subset B_1$.

3. a) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2$. Let $A_1 = \{-1, 2\}$ and $B_1 = \{-9, 0, 4\}$. Determine $f^{-1}[f[A_1]]$ and $f[f^{-1}[B_1]]$.

b) Part **a)** shows that the reverse subset directions in Question **2.** do not hold. However, each of these can be proved in the reverse direction with one additional hypothesis condition. Determine the needed additional condition on f for each of the subset statements that will make the reverse subset inclusion statement true. Prove your claims.

4. Suppose $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ are bijections, and that $A_1 \cap A_2 = B_1 \cap B_2 = \emptyset$. Define $f : A_1 \cup A_2 \to B_1 \cup B_2$ by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2. \end{cases}$$

Prove that f is a bijection from $A_1 \cup A_2$ to $B_1 \cup B_2$.

5. Prove that there is no bijection between a set A and its power set $\mathcal{P}(A)$.