MATH 103 – Homework Assignment II

1. For each $r \in \mathbb{Q}^+$, let

$$A_r = (1/2, 1 + r)$$

be the open interval of real numbers with endpoints $1/2$ and $1 + r$. Find the following sets:

a) $\bigcup_{r \in \mathbb{Q}^+} A_r$

b) $\bigcap_{r \in \mathbb{Q}^+} A_r$

c) $\bigcup_{r \in \mathbb{Q}^+} (\mathbb{R} \setminus A_r)$

d) $\bigcap_{r \in \mathbb{Q}^+} (\mathbb{R} \setminus A_r)$

2. For a set A, the *power set* of A is denoted by $\mathcal{P}(A)$; it is the set of *all* subsets of A:

$$\mathcal{P}(A) = \{X | X \subseteq A\}.$$

Prove that for all sets A and B,

$$\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B).$$

3. By using induction, prove that for every $n \in \mathbb{Z}^+$, either $n = 2k$ or $n = 2k + 1$ for some $k \in \mathbb{Z}^+$. Use this result to show that the statement is true for all $n \in \mathbb{Z}$.

4. Prove by *principal of mathematical induction* that for all $n \in \mathbb{Z}^+$,

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1) = (x - 1)(\sum_{k=0}^{n-1} x^k)$$

for all $x \in \mathbb{R}$.

5. Let $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the relation $\sim$ on A as follows: For any $a, b \in A$, define

$$a \sim b \Leftrightarrow b = ka \text{ for some } k \in \mathbb{Z}^+.$$

a) Prove that $\sim$ is a partial ordering on A.

b) Draw the lattice diagram for this partial ordering $\sim$ on A.

c) Is $\sim$ a total ordering on A? Explain.

d) What are the maximal and minimal elements of A?

e) Does A have greatest and least element? If it has, what are they?