IUE – MATH 103 – Fundamentals of Mathematics

2^{nd} Midterm — December 19, 2013 — 16:00 - 17:50

Name Surname: ______________________ **ID #:** ______________________

Q1	Q2	Q3	Q4	TOTAL
25	25	25	25	100

Important: Show all your work. Answers without sufficient explanation might not get full credit. Be neat.

Signature: ______________________

GOOD LUCK!

1. a) State the *Fundamental Theorem of Arithmetic.*
b) Write the definition of a function.

Solution: a) Any integer greater than 1 can be written uniquely as a product of prime numbers, up to order of the product.

b) A function f from A to B is a relation, $f \subseteq A \times B$ such that

i) $\forall x \in A \; \exists y \in B$ such that $(x, y) \in f$ [or $f(x) = y$]

ii) $\forall x \in A$, if $(x, y_1) \in f$ & $(x, y_2) \in f$ then $y_1 = y_2$ [image is unique.]

2. Check whether each of the following relations is an equivalence relation or a partial order, and/or a total order, and/or well order, or none. Depending on your answer, also find or describe, if any, greatest, least, maximal, minimal elements and/or equivalence classes with the corresponding partition.

a) $A = \{1, 2, 3\} \quad R_1 = \{(1, 1), (2, 1), (3, 1)\}$.

b) $B = \{2, 3, 6\} \quad R_2 = \{(a, b) \mid \gcd(a, b) \neq 1\}$.

c) $C = \mathcal{P}(\mathbb{N}) \quad X, Y \in R_3$ iff $X \cap Y \neq \emptyset$, where $\mathcal{P}$ denotes the power set.

d) $D = \mathbb{R} \quad (x, y) \in R_4$ iff $xy > 0 \vee x^2 + y^2 = 0$.

Solution: a) R_1 is neither reflexive ($(2,2) \notin R_1$) nor irreflexive ($(1,1) \in R_1$). Therefore, R_1 is none of the above mentioned reliations.

b) $R_2 = \{(2, 2), (3, 3), (6, 6), (2, 6), (3, 6)\}$

R_2 is clearly a partially order and not a total order.

Greatest element=Maximal Element=6

No least element. Minimal elements=$\{2, 3\}$.

c) R_3 is not reflexive since $(\emptyset, \emptyset) \notin R_3$. Therefore, R_3 is none of the above.

d) R_4 is reflexive. $\quad x.x = x^2 \geq 0 \to x^2 > 0$ or $2x^2 = 0$.

symmetric. $\quad xy > 0 \to yx > 0$ or $x^2 + y^2 = 0 \to y^2 + x^2 = 0$

transitive.

$$\begin{cases} xy > 0 \\ yz > 0 \end{cases} \to xzy^2 > 0 \to xz > 0$$

or

$$\begin{cases} x^2 + y^2 = 0 \to (x, y) = (0, 0) \\ y^2 + z^2 = 0 \to (y, z) = (0, 0) \end{cases} \to x = y = z = 0 \to x^2 + y^2 = 0$$

So, R_4 is an equivalence relation

$[1]$ =positive numbers

$[0] = \{0\}$

$[-1]$ =Negative numbers.

$R = (-\infty, 0) \cup [0, 0] \cup (0, \infty)$

3. Let $f : A \to B$ be function. Then, prove or disprove the following statements.
a) $f[X \cap Y] = f[X] \cap f[Y]$ for any $X, Y \subseteq A$.
b) $f^{-1}[Z^c] = (f^{-1}[Z])^c$ for any $Z \subseteq B$.

Solution: a) Disprove:
$A = \{1, 2\}$, $B = \{3\}$, $f(x) = 3$.
$X = \{1\}$ and $Y = \{2\}$. $X \cap Y = \emptyset$ and $f[X \cap Y] = f[\emptyset] = \emptyset$.
But $f[X] = \{3\}$ and $f[Y] = \{3\} \to f[X] \cap f[Y] = \{3\} \neq \emptyset$.

b) $\subseteq$:
Let $x \in f^{-1}[Z^c]$. Then $f(x) = y \in Z^c \to y \notin Z \to f^{-1}[y] \nsubseteq f^{-1}[Z] \wedge x \in f^{-1}(y) \to x \notin f^{-1}[Z] \to x \in \left(f^{-1}[Z]\right)^c$
$\supseteq$:
Let $a \in \left(f^{-1}[Z]\right)^c$. $a \notin f^{-1}[Z]$ if and only if $b = f(a) \notin Z \to b \in Z^c \to a \in f^{-1}(b) \subseteq f^{-1}[Z^c] \to a \in f^{-1}[Z^c]$

4. Prove by Principal of Mathematical Induction that for $a_1, \dots, a_n \in \mathbb{R}$

$$\left|\sum_{k=1}^{n} a_k\right| \leq \sum_{k=1}^{n} |a_k|.$$

Solution: Initial Step: $n = 1$

$$\left.\begin{array}{l} \left|\displaystyle\sum_{k=1}^{n} a_k\right| = |a_1| \\ \displaystyle\sum_{k=1}^{n} |a_k| = |a_1| \end{array}\right\} \leq$$

Induction Step: Suppose that the inequality holds for n. Then for $n+1$ we have

$$\begin{aligned} \left|\sum_{k=1}^{n+1} a_k\right| &= \left|\sum_{k=1}^{n} a_k\right| + a_{n+1} \\ &\leq \left|\sum_{k=1}^{n} a_k\right| + |a_{n+1}| \quad \text{by usual triangle inequality} \\ &\leq \sum_{k=1}^{n} |a_k| + |a_{n+1}| \qquad \text{by induction hyptohesis} \\ &= \sum_{k=1}^{n+1} |a_k| \end{aligned}$$

Conclusion: The inequality for $a_i \in R$

$\left|\displaystyle\sum_{k=1}^{n} a_k\right| \leq \displaystyle\sum_{k=1}^{n} |a_k|$ is valid for $\forall n \geq 1$.