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# A note on large deviations for weak records

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**Abstract**

Let $X^w(n)$ be weak record values derived from samples consisting of independent identically distributed discrete random variables. A limit theorem for large deviations for $X^w(n)$ is proposed in the present paper.


*MSC:* 60G70; 62G30

*Keywords:* Central limit theorem; Large deviations; Weak records

## 1. Introduction

In this paper, we assume that $X_1, X_2, \ldots$ are independent identically distributed random variables taking non-negative integer values with the distribution $F(n) = P\{X_1 \leqslant n\}$ satisfying the condition $F(n) < 1$ ($\forall\ n \geqslant 0$).

The notations of weak record times $L^w(n)$ ($n \geqslant 1$) and weak record values $X^w(n)$ ($n \geqslant 1$) were proposed by Vervaat (1973) as follows:

$$L^w(1) = 1, \quad L^w(n+1) = \min\{j : j > L^w(n),\ X_j \geqslant X_{L^w(n)}\},$$
$$X^w(n) = X_{L^w(n)} \quad (n \geqslant 1).$$

Weak records have been studied later by Stepanov (1992, 1993), Aliev (1998, 1999), López-Blázquez and Wesołowski (2001), Wesołowski and Ahsanullah (2001), Stepanov et al. (2003), Wesołowski and López-Blázquez (2004), Danielak and Dembińska (2006), Dembińska and López-Blázquez (2005), and Dembińska and Stepanov (2006). The above-mentioned papers have derived different limit and characterization results for weak records. Some topics related to weak records are also presented in the books of Arnold et al. (1998) and Nevzorov (2001).

It should be noted that the direct approach for producing limit theorems for weak record values does not work here, because weak record values are dependent random variables. However, instead of considering dependent weak records, one can study sums of independent geometrically distributed random variables.

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Define variables $\xi_i^w$ $(i = 0, 1, \ldots)$ by

$\xi_i^w = k$ if there are exactly $k$ weak record values that are equal to $i$.

**Lemma 1.1.** *The variables* $\xi_i^w$ $(i = 0, 1, \ldots)$ *are independent and*

$$P\{\xi_i^w = k\} = \beta_i(1-\beta_i)^k \quad (k = 0, 1, \ldots, i = 0, 1, \ldots), \tag{1.1}$$

*where* $\beta_i = q_{i+1}/q_i$ *and* $q_n = P\{X_1 \geqslant n\}$.

**Representation 1.1.** *The following equality holds true*

$$P\{X^w(n) > m\} = P\{\xi_0^w + \xi_1^w + \cdots + \xi_m^w < n\} \quad \textit{for } n \geqslant 1, \; m \geqslant 0.$$

Lemma 1.1 and Representation 1.1 have been proposed in Stepanov (1992). The strong law of large numbers, the strong law of iterated logarithm and the central limit theorem for weak records have been obtained in that paper by virtue of Lemma 1.1 and Representation 1.1. Define functions $R(n)$ and $B(n)$ by

$$R(n) = \sum_{i=0}^{n} \frac{1-\beta_i}{\beta_i}, \quad B(n) = \sum_{i=0}^{n} \frac{1-\beta_i}{\beta_i^2} \quad (n \geqslant 0).$$

The central limit theorem for $R(X^w(n))$ has the following form.

**Theorem 1.1.** *Let F satisfy the conditions*

$$a = \inf_{n \geqslant 0} \beta_n > 0. \tag{1.2}$$

$$\lim_{n\to\infty} \frac{R(n)}{B(n)} = \varepsilon \in [a, 1]. \tag{1.3}$$

*Then*

$$\sqrt{\varepsilon}\,\frac{R(X^w(n)) - n}{\sqrt{n}} \to_d \xi_{N(0,1)} \quad (n \to \infty),$$

*where* $\xi_{N(0,1)}$ *is a random variable having the Normal Law with parameters* 0 *and* 1.

In view of Theorem 1.1 it is interesting to study the zone of the normal convergence for $R(X^w(n))$. It is also important, because at the moment we do not know any results of this kind.

Our paper has the following continuation. The result on large deviations for $X^w(n)$ is presented in Section 2. Some examples illustrating this result are proposed in Section 3.

## 2. Results

**Theorem 2.1.** *Let* $\beta_n \to \beta \in (0, 1)$ *and* $x \geqslant 0$, $x = \mathrm{o}(\sqrt{n})$. *Then*

$$\frac{P\{(1-\beta)X^w(n) > x\sqrt{\beta n} + \beta n\}}{1 - \Phi(x)} = \exp\left\{-\frac{x^3\sqrt{1-\beta}}{\sqrt{\beta n}}\lambda_n\left(-\frac{x}{\sqrt{n}}\right)\right\}\left[1 + \mathrm{O}\left(\frac{x+1}{\sqrt{n}}\right)\right] \tag{2.1}$$

$$\frac{P\{(1-\beta)X^w(n) < -x\sqrt{\beta n} + \beta n\}}{\Phi(-x)} = \exp\left\{\frac{x^3\sqrt{1-\beta}}{\sqrt{\beta n}}\lambda_n\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1 + \mathrm{O}\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{2.2}$$

*where $\Phi(x)$ is the distribution function of the standard normal random variable and*

$$\lambda_n(t) = \sum_{k=0}^{\infty} a_{kn} t^k \tag{2.3}$$

*is a power series that for all large enough n is majorized by another power series which converges in some circle and its coefficients do not depend on n. The series $\lambda_n(t)$ for all small enough $|t|$ converges uniformly with respect to n.*

**Proof of Theorem 2.1.** Let us make use of Theorem 2.2 proved in Petrov (1968) (see also Petrov (1975)). □

**Theorem 2.2.** *Let $\tilde{\xi}_n$ $(n \geqslant 0)$ be independent variables with zero means. Let positive constants $g, G, H$ and $\delta$ exist such that the inequalities $g \leqslant |Ee^{z\tilde{\xi}_n}| \leqslant G$, $B(n) \geqslant (n+1)\delta$ $(n \geqslant 0)$ hold in the circle $|z| < H$. Then for $x \geqslant 0$, $x = o(\sqrt{n})$*

$$\frac{P\{\widetilde{S}(n) > x\sqrt{B(n)}\}}{1-\Phi(x)} = \exp\left\{\frac{x^3}{\sqrt{n}}\widetilde{\lambda}_n\left(\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{2.4}$$

$$\frac{P\{\widetilde{S}(n) < -x\sqrt{B(n)}\}}{\Phi(-x)} = \exp\left\{-\frac{x^3}{\sqrt{n}}\widetilde{\lambda}_n\left(-\frac{x}{\sqrt{n}}\right)\right\}\left[1 + O\left(\frac{x+1}{\sqrt{n}}\right)\right], \tag{2.5}$$

*where $\widetilde{S}(n) = \widetilde{\xi}_0 + \cdots + \widetilde{\xi}_n$, $B(n) = \sum_{i=0}^{n} Var\,\widetilde{\xi}_i$ and $\widetilde{\lambda}_n(t) = \sum_{k=0}^{\infty}\widetilde{a}_{kn}t^k$ is a power series that for all large enough n is majorized by another power series which converges in some circle and its coefficients do not depend on n. The series $\widetilde{\lambda}_n(t)$ for all small enough $|t|$ converges uniformly with respect to n. The coefficients $\widetilde{a}_{kn}$ for any k can be expressed through all the cumulants $(1/i^j)[\frac{d^j}{dt^j}\log f_{\tilde{\xi}_m}(t)](m = 0, \ldots n;\ j = 1, \ldots, k+3)$, where $f_{\tilde{\xi}_m}(t)$is the characteristic function of $\widetilde{\xi}_m$.*

Returning back to the proof of Theorem 2.1, let us denote

$$\widetilde{\xi}_n = \xi_n^w - \frac{1-\beta_n}{\beta_n} \quad (n \geqslant 0). \tag{2.6}$$

Then

$$E\widetilde{\xi}_n = 0, \quad Ee^{z\tilde{\xi}_n} = \frac{\beta_n e^{(-(1-\beta_n)/\beta_n)z}}{1-(1-\beta_n)e^z} \quad (n \geqslant 0).$$

Since $\beta_n > 0$ $(n \geqslant 0)$ and $\beta_n \to \beta \in (0,1)$, constants $c, b$ exist $(0 < c < b < 1)$ such that $c < \beta_n < b$ for $n \geqslant 0$. Choose $H$ satisfying the condition $0 < H < -\ln(1-c)$. Then, for $z$ those which belong to the circle $|z| < H$, the equalities hold

$$e^{-H} < |e^z| < e^H, \quad e^{-H(1-\beta_n)/\beta_n} < |e^{(-(1-\beta_n)/\beta_n)z}| < e^{H(1-\beta_n)/\beta_n}.$$

Observe that if $|z| < 1$ then $|1-z| > 1-|z|$. Consequently,

$$|1-(1-\beta_n)e^z| > 1-(1-c)e^H > 0.$$

From the above suggestions we obtain the following bounds

$$g < |Ee^{z\tilde{\xi}_n}| < G,$$

where $g = ce^{-H/c}/(1+e^H)$, $G = e^{H/c}/(1-(1-c)e^H)$. It is obvious that

$$B(n) > \sum_{i=0}^{n}(1-\beta_n) > \delta(n+1),$$

where $\delta = (1-b)$. In this way, the conditions of Theorem 2.2 are fulfilled for variables $\widetilde{\xi}_n$ which were defined by (2.6). Then for $\widetilde{S}(n) = \widetilde{\xi}_0 + \cdots + \widetilde{\xi}_n$ equalities (2.4) and (2.5) hold.

Since

$$R(n) = \frac{1-\beta}{\beta}n + \mathrm{O}(1), \quad B(n) = \frac{1-\beta}{\beta^2}n + \mathrm{O}(1) \quad (n \to \infty),$$

the left-hand side of (2.4) is equal to

$$\frac{P\{S(n) > ((1-\beta)/\beta)n + x((\sqrt{1-\beta})/\beta)\sqrt{n}\}}{1-\Phi(x)} + \mathrm{o}(1) \quad (n \to \infty).$$

Put $m = [(1-\beta)/\beta)n + x((\sqrt{1-\beta})/\beta)\sqrt{n}]$, where $[z]$ means the integral part of the number $z$. Observe that for all large enough $n$ and $m$

$$n \sim \frac{\beta m}{1-\beta}, \quad n = \frac{\beta m}{1-\beta} - x\frac{\sqrt{\beta m}}{1-\beta} + \mathrm{O}(1).$$

Applying Representation 1.1 and using the equality $1-\Phi(x) = \Phi(-x)$, one can get (2.2). The coefficients of the series $\lambda_n(t)$ in (2.2) are the following

$$a_{k,m} = \widetilde{a}_{k,[\beta m/(1-\beta) - x(\sqrt{\beta m}/(1-\beta)) + \mathrm{O}(1)]}\left(\frac{1-\beta}{\beta}\right)^{k/2}.$$

The same way from (2.5) one can get (2.1). □

## 3. Examples

**Example 3.1.** Let $X_1$ be geometrically distributed random variable with parameter $p \in (0,1)$. Then

$$\beta_n = q, \quad R(n) = \frac{(1-q)(n+1)}{q}, \quad B(n) = \frac{(1-q)(n+1)}{q^2},$$

Condition (1.2) and (1.3) are met here and

$$\frac{(1-q)X^w(n) - nq}{\sqrt{qn}} \to_d \xi_{N(0,1)} \quad (n \to \infty).$$

The conditions of Theorem 2.1 are fulfilled and $0 < x < \mathrm{o}(\sqrt{n})$ is the zone of normality for $(1-q)X^w(n)$.

**Example 3.2.** Let us consider the discrete logarithmic distribution

$$P\{X_1 = k\} = \frac{1}{[-\log(1-a)]} \cdot \frac{a^{k+1}}{k+1} \quad (k \geqslant 0,\ 0 < a < 1).$$

We have

$$\beta_n = \frac{\sum_{k=n+2}^{\infty}(a^k/k)}{\sum_{k=n+1}^{\infty}(a^k/k)}.$$

Since $(n+1)/(n+2) < (n+2)/(n+3) < (n+3)/(n+4) < \cdots$, the following inequality $a(n+1)/(n+2) < \beta_n < a$ holds. Then for all large enough $n$

$$R(n) \sim \frac{1-a}{a}n, \quad B(n) \sim \frac{1-a}{a^2}n.$$

The conditions of Theorem 1.1 hold and

$$\frac{(1-a)X^w(n) - na}{\sqrt{an}} \to_d \xi_{N(0,1)} \quad (n \to \infty).$$

Theorem 2.1 is also valid here and $0 < x < \mathrm{o}(\sqrt{n})$ is the zone of normality for $(1-a)X^w(n)$.

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