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ELSEVIER
Statistics & Probability Letters 63 (2003) 123–131
STATISTICS & PROBABILITY LETTERS
www.elsevier.com/locate/stapro

On a new sample rank of an order statistics and its concomitant

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Received August 2001; received in revised form September 2002

Abstract

The marginal and joint distributions of the new sample rank of an order statistic and its concomitant are derived.
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MSC: 62G30; 62F07

Keywords: Order statistics; Rank; Concomitant

1. Introduction

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be a random sample from a continuous bivariate distribution function (d.f.) $F_{X,Y}(x, y)$. If we denote by $X_{r:n}$ the rth-order statistic of the X sample values then the Y values associated with $X_{r:n}$ is called the concomitant of the rth-order statistic and is denoted by $Y_{[r:n]}$. The concomitants are of interest in selection and prediction problems. For example, when $k(< n)$ individuals having the highest X-scores are selected, we may wish to know the behavior of the corresponding Y-scores. For a detailed review of concomitants see Bhattacharya (1984) and David (1993).

Let $f(y \mid x)$ be the conditional density function of Y given $X = x$ and $f_{r:n}(x)$ is the probability density function (p.d.f.) of $X_{r:n}$. Then the p.d.f. of the rth concomitant $Y_{[r:n]}$ is

$$f_{[r:n]}(y) = \int_{-\infty}^{\infty} f(y \mid x) f_{r:n}(x)\, \mathrm{d}x$$

(see David, 1981).

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0167-7152/03/$ - see front matter © 2003 Elsevier Science B.V. All rights reserved.
doi:10.1016/S0167-7152(03)00059-2

The joint p.d.f. of the rth-order statistic $X_{r:n}$ and its concomitant $Y_{[r:n]}$ is

$$f_{X_{r:n},Y_{[r:n]}}(x,y) = f(y\,|\,x) f_{r:n}(x).$$

David et al. (1977) have considered the rank of $Y_{[r:n]}$ among the nY_i's, i.e.

$$R_{r,n} = \sum_{i=1}^{m} J(Y_{[r:n]} - Y_i),$$

where $J(x) = 1$, if $x \geqslant 0$ and $J(x) = 0$, if $x < 0$ and derived the distribution of $R_{r,n}$.

Let $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), \ldots, (X_{n+m}, Y_{n+m})$ be a random sample with a continuous d.f. $G(x,y)$ and independent of $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. Denote by $X_{r:n}, Y_{[r:n]}$ the rth-order statistic and its concomitant, respectively. In this paper, we consider the marginal and joint distributions of the rank of $X_{r:n}$ and $Y_{[r:n]}$ among the new sample $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), \ldots, (X_{n+m}, Y_{n+m})$.

2. Distributions of the new sample ranks of $X_{r:n}$ and $Y_{[r:n]}$

Let (X_i, Y_i), $i = 1, 2, \ldots, n$ be a random sample from a continuous bivariate d.f. $F(x,y)$. Denote by $X_{r:n}, Y_{[r:n]}$ the rth-order statistic and its concomitant, respectively. Suppose $(X_{n+1}, Y_{n+1}), (X_{n+2}, Y_{n+2}), \ldots, (X_{n+m}, Y_{n+m})$ $(m \geqslant 1)$ is another random sample with a continuous d.f. $G(x,y)$ and independent of $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$. For $1 \leqslant r \leqslant n$ and $m \geqslant 1$ let us define the following random variables:

$$\eta_1 = \sum_{i=1}^{m} I(X_{r:n} - X_{n+i}),$$

$$\eta_2 = \sum_{i=1}^{m} I(Y_{[r:n]} - Y_{n+i}),$$

where $I(x) = 1$, if $x > 0$ and $I(x) = 0$, if $x \leqslant 0$ (actually η_i, $i = 1, 2$ depends on r, n and m, but we suppress them in our notations to avoid confusion).

The random variable $\xi_1 = \eta_1 + 1$ shows the rank of $X_{r:n}$ among the $X_{n+1}, X_{n+2}, \ldots, X_{n+m}$ and the random variable $\xi_2 = \eta_2 + 1$ shows the rank of $Y_{[r:n]}$ among the $Y_{n+1}, Y_{n+2}, \ldots, Y_{n+m}$.

Denote by (X_i, Y_i) the two tests results taken by the ith individual. For an individual from the first sample, whose results are $(X_{r:n}, Y_{[r:n]})$, what is the probability of the event that the individual's first test rank will be k and the second test rank will be l among the second sample? To give an answer to this question we have to know the joint distribution of the random variables ξ_1 and ξ_2. The marginal distributions of ξ_1 and ξ_2 may also be of interest.

Denote by $F_X(x), F_Y(x), G_X(x), G_Y(x)$ the corresponding marginals of $F(x,y)$ and $G(x,y)$. The finite and asymptotic distributions for large m of the random variable η_1 are well studied in literature. It is well known that if $F_X(x)$ and $G_X(x)$ are continuous distribution functions then under hypothesis $\mathrm{H}_0 : F_X = G_X$ the finite distribution of η_1 is a negative hypergeometric distribution of the first kind, i.e.

$$P\{\eta_1 = k\} = \frac{\binom{r+k-1}{r-1}\binom{m+n-k-r}{n-r}}{\binom{m+n}{n}}, \quad k = 0, 1, \ldots, m. \tag{1}$$

Note that many authors have been interested in the exceedance statistic η_1. For more details one can see the classical result of Gumbel and Von Schelling (1950), see also Gumbel (1954a, b), Epstein (1954) and Sarkadi (1957). Katzenbeisser (1985, 1986), Matveychuk and Petunin (1990, 1991) and Johnson and Kotz (1991, 1994) have used the statistic η_1 for the construction of test criterion for testing hypothesis H_0 against various class of alternatives. Wesolowski and Ahsanullah (1998) provided some interesting results for the distribution of η_1 in case of different F_X and G_X. Recently, Bairamov and Kotz (2001) derived the distribution of η_1 when the underlying distribution is arbitrary, i.e. when F_X contains continuous, discrete and singular components simultaneously.

Evidently, it follows from (1) that

$$P\{\xi_1 = k\} = P\{\eta_1 = k-1\} = \frac{\binom{r+k-2}{r-1}\binom{m+n-k-r+1}{n-r}}{\binom{m+n}{n}}, \quad k = 1, 2, \ldots, m+1.$$

Below, we provide the marginal distribution of ξ_2.

One can write

$$\begin{aligned} P\{\xi_2 = l\} &= P\{\eta_2 = l-1\} \\ &= \binom{m}{l-1} P\{Y_{n+1} < Y_{[r:n]}, \ldots, Y_{n+l-1} < Y_{[r:n]}, Y_{n+l} > Y_{[r:n]}, \ldots, Y_{n+m} > Y_{[r:n]}\} \end{aligned} \tag{2}$$

by conditioning on $Y_{[r:n]}$, we then have from (2)

$$\begin{aligned} &P\{\xi_2 = l\} \\ &= \binom{m}{l-1} \int_{-\infty}^{\infty} (G_Y(y))^{l-1}(1 - G_Y(y))^{m-l+1} f_{[r:n]}(y)\,\mathrm{d}y \\ &= \binom{m}{l-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G_Y(y))^{l-1}(1 - G_Y(y))^{m-l+1} f(y\,|\,x) f_{r:n}(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \binom{m}{l-1} \frac{1}{B(r, n-r+1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(G_Y(y))^{l-1}(1 - G_Y(y))^{m-l+1} \\ &\quad \times (F_X(x))^{r-1}(1 - F_X(x))^{n-r} f(x, y)]\,\mathrm{d}x\,\mathrm{d}y, \quad l = 1, 2, \ldots, m+1, \end{aligned} \tag{3}$$

where $B(a, b)$ is a Beta function.

Table 1
$P\{\xi_2 = l\}$ as a function of α

n	r	α	m	l	$P\{\xi_2 = l\}$	n	r	α	m	l	$P\{\xi_2 = l\}$
5	2	0.1	1	1	0.506	5	2	−0.1	1	1	0.494
5	2	0.1	1	2	0.494	5	2	−0.1	1	2	0.506
10	4	0.5	2	1	0.356	10	4	−0.5	2	1	0.311
10	4	0.5	2	2	0.333	10	4	−0.5	2	2	0.333
10	4	0.5	2	3	0.311	10	4	−0.5	2	3	0.356
10	5	0.9	3	1	0.262	10	5	−0.9	3	1	0.238
10	5	0.9	3	2	0.254	10	5	−0.9	3	2	0.246
10	5	0.9	3	3	0.246	10	5	−0.9	3	3	0.254
10	5	0.9	3	4	0.238	10	5	−0.9	3	4	0.262

To illustrate the distribution of ξ_2, let us consider the bivariate Farlie–Gumbel–Morgenstern (FGM) distribution whose d.f. and p.d.f. are given, respectively:

$$F(x,y) = F(x)F(y)\{1 + \alpha(1 - F(x))(1 - F(y))\},$$

$$f(x,y) = f(x)f(y)\{1 + \alpha(1 - 2F(x))(1 - 2F(y))\}, \quad -1 \leqslant \alpha \leqslant 1, \ -\infty < x, y < \infty. \tag{4}$$

The reason of choosing FGM distribution is a simple analytical form of this distribution which allows us to see the distribution of ξ_2 in a simple form.

Taking $F(x,y) = G(x,y)$, we have from (3)

$$\begin{aligned} P\{\xi_2 = l\} &= \binom{m}{l-1} \frac{1}{B(r,n-r+1)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [(F(y))^{l-1}(1-F(y))^{m-l+1}(F(x))^{r-1}(1-F(x))^{n-r} \\ &\quad \times f(x)f(y)\{1 + \alpha(1-2F(x))(1-2F(y))\}]\,\mathrm{d}x\,\mathrm{d}y \\ &= \binom{m}{l-1} \frac{1}{B(r,n-r+1)} \int_0^1\int_0^1 v^{l-1}(1-v)^{m-l+1}u^{r-1}(1-u)^{n-r} \\ &\quad \times \{1 + \alpha(1-2u)(1-2v)\}\,\mathrm{d}u\,\mathrm{d}v \\ &= \binom{m}{l-1} \frac{1}{B(r,n-r+1)} \{B(r,n-r+1)B(l,m-l+2) \\ &\quad + \alpha[B(l,m-l+2) - 2B(l+1,m-l+2)][B(r,n-r+1) \\ &\quad - 2B(r+1,n-r+1)]\}, \quad l = 1,2,\ldots,m+1, \ 1 \leqslant r \leqslant n. \end{aligned}$$

Some numerical results for $P\{\xi_2 = l\}$ are provided in Table 1.

It is interesting that when m tends to infinity under continuity assumptions on underlying distribution functions asymptotic distribution of η_1/m is $P\{G_X(X_{r:n}) \leqslant x\}$ which turns into Beta$(r, n-r+1)$ distribution under hypothesis H_0 (see Matveychuk and Petunin (1991) and for a different proof and related record models see Bairamov (1997), Wesolowski and Ahsanullah (1998) and Bairamov and Eryilmaz (2000)). Using the same technique presented in the proof of Theorem 3.1 of Bairamov (1997) it is not difficult to observe that under continuity assumptions of underlying distributions the asymptotic distribution of η_2/m for large m is $P\{G_Y(Y_{[r:n]}) \leqslant x\}$.

Joint distribution of ξ_1 and ξ_2: Our main goal is to find the joint distribution of the random variables ξ_1 and ξ_2. The compound event $\{\xi_1 = k, \xi_2 = l\} \equiv \{\eta_1 = k-1, \eta_2 = l-1\}$ can occur is best identified with the following table.

	$X_{n+i} < X_{r:n}$	$X_{n+i} > X_{r:n}$	
$Y_{n+i} < Y_{[r:n]}$	j	$l-j-1$	$l-1$
$Y_{n+i} > Y_{[r:n]}$	$k-j-1$	$m-k-l+j+2$	$m-l+1$
	$k-1$	$m-k+1$	m

Denote $A_i = \{X_{n+i} < X_{r:n}, Y_{n+i} < Y_{[r:n]}\}$, $B_i = \{X_{n+i} > X_{r:n}, Y_{n+i} < Y_{[r:n]}\}$, $C_i = \{X_{n+i} < X_{r:n}, Y_{n+i} > Y_{[r:n]}\}$, $D_i = \{X_{n+i} > X_{r:n}, Y_{n+i} > Y_{[r:n]}\}$. (Since underlying distributions are continuous and the contribution of the probabilities of the events that occur with $X_{n+i} = X_{r:n}$ or $Y_{n+i} = Y_{[r:n]}$, $i = 1, 2, \ldots, m$ is zero, to avoid complicated expressions we do not include these events to the scheme.) We have

$$P\{\xi_1 = k, \xi_2 = l\} = \sum_{j=\max(0,k+l-m-2)}^{\min(k-1,l-1)} F_{m,j,k,l} P\{A_{i_1} A_{i_2} \ldots A_{i_j} B_{i_{j+1}} B_{i_{j+2}} \ldots B_{i_{l-1}} C_{i_l} C_{i_{l+1}} \ldots C_{i_{k-j+l-2}} D_{i_{k-j-1}} \ldots D_{i_m}\} \tag{5}$$

where $F_{m,j,k,l} = m!/(j!(l-j-1)!(k-j-1)!(m-k-l+j+2)!)$.

By conditioning on $X_{r:n}, Y_{[r:n]}$, we then have from (5)

$$P\{\xi_1 = k, \xi_2 = l\} = \sum_{j=\max(0,k+l-m-2)}^{\min(k-1,l-1)} F_{m,j,k,l} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\pi_1(x,y))^j (\pi_2(x,y))^{l-j-1} (\pi_3(x,y))^{k-j-1} \times (\pi_4(x,y))^{m-k-l+j+2} f(y \mid x) f_{r:n}(x) \, dx \, dy, \quad k, l = 1, 2, \ldots, m+1,$$

where

$$\pi_1(x,y) = P\{X_{n+1} < x, Y_{n+1} < y\},$$

$$\pi_2(x,y) = P\{X_{n+1} > x, Y_{n+1} < y\},$$

Table 2
$P\{\xi_1 = k, \xi_2 = l\}$ as a function of α

n	r	α	m	k	l	$P\{\xi_1 = k, \xi_2 = l\}$
5	2	0.1	1	1	1	0.341
5	2	0.1	1	1	2	0.325
5	2	0.1	1	2	1	0.165
5	2	0.1	1	2	2	0.169
10	4	0.5	2	1	1	0.166
10	4	0.5	2	1	2	0.140
10	4	0.5	2	2	1	0.145
10	4	0.5	2	2	2	0.143
10	4	0.5	2	2	3	0.136
10	4	0.5	2	3	2	0.050
10	4	0.5	2	3	1	0.045
10	4	0.5	2	1	3	0.118
10	4	0.5	2	3	3	0.057

$$\pi_3(x,y) = P\{X_{n+1} < x, Y_{n+1} > y\},$$

$$\pi_4(x,y) = P\{X_{n+1} > x, Y_{n+1} > y\}.$$

Let us consider the FGM distribution given by (5), in this case

$$\pi_1(x,y) = F(x)F(y)\{1+\alpha(1-F(x))(1-F(y))\},$$

$$\pi_2(x,y) = F(y) - F(x)F(y)\{1+\alpha(1-F(x))(1-F(y))\},$$

$$\pi_3(x,y) = F(x) - F(x)F(y)\{1+\alpha(1-F(x))(1-F(y))\},$$

$$\pi_4(x,y) = 1 - F(x) - F(y) + F(x)F(y)\{1+\alpha(1-F(x))(1-F(y))\},$$

hence,

$$P\{\xi_1 = k, \xi_2 = l\}$$

$$= \frac{1}{B(r,n-r+1)} \sum_{j=\max(0,k+l-m-2)}^{\min(k-1,l-1)} F_{m,j,k,l} \int_0^1 \int_0^1 [uv\{1+\alpha(1-u)(1-v)\}]^j$$

$$\times [v - uv\{1+\alpha(1-u)(1-v)\}]^{l-j-1} [u - uv\{1+\alpha(1-u)(1-v)\}]^{k-j-1}$$

$$\times [1 - u - v + uv\{1+\alpha(1-u)(1-v)\}]^{m-k-l+j+2}$$

$$\times \{1+\alpha(1-2u)(1-2v)\} u^{r-1}(1-u)^{n-r} \, du \, dv, \quad k,l = 1,2,\ldots,m+1.$$

In Table 2, we provide some numerical results for $P\{\xi_1 = k, \xi_2 = l\}$.

From Table 2, one can check that for $n = 5$, $r = 2$, $\alpha = 0.1$, $m = 1$

$$P\{\xi_2 = 2\} = P\{\xi_1 = 1, \xi_2 = 2\} + P\{\xi_1 = 2, \xi_2 = 2\} = 0.325 + 0.169 = 0.494,$$

which coincides with the result in Table 1.

2.1. Numerical example

It is said that complex hydrological events such as floods and storms always appear to be multivariate events that are characterized by a few correlated random variables. Flood volume and flood duration are two flood characteristics. Bivariate FGM-type distributions are used to represent the joint distribution of these variables. (Yue et al., 2001)

Example. Suppose that X_i and Y_i represent the flood volume and flood duration in a certain locality for the ith year ($i = 1, 2, \ldots$). Suppose that the joint distribution of these two random variables is FGM-type distribution with $\alpha = 0.9$. Under these assumptions what is the probability that the largest flood volume and corresponding flood duration during the past 10 years will be exceeded at least once during the next 10 years?

It is easy to see that the required probability is

$$p = \sum_{k=2}^{11} \sum_{l=2}^{11} P\{\xi_1 = k, \xi_2 = l\} = 0.94.$$

$(m = 10,\ r = n = 10)$.

2.2. The correlation between ξ_1 and ξ_2

One can write for the variance of ξ_1

$$\begin{aligned}
\mathrm{Var}(\xi_1) = \mathrm{Var}(\eta_1) &= \sum_{i=1}^{m} P\{X_{n+i} < X_{r:n}\} \\
&\quad + 2\sum_{i<j} P\{X_{n+i} < X_{r:n}, X_{n+j} < X_{r:n}\} - \left(\sum_{i=1}^{m} P\{X_{n+i} < X_{r:n}\}\right)^2 \\
&= m\int_{-\infty}^{\infty} G_X(x) f_{r:n}(x)\,\mathrm{d}x + 2\,\frac{m(m-1)}{2}\int_{-\infty}^{\infty} (G_X(x))^2 f_{r:n}(x)\,\mathrm{d}x \\
&\quad - \left(m\int_{-\infty}^{\infty} G_X(x) f_{r:n}(x)\,\mathrm{d}x\right)^2
\end{aligned}$$

and the variance of ξ_2

$$\begin{aligned}
&\mathrm{Var}(\xi_2) \\
&= m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_Y(y) f(y\,|\,x) f_{r:n}(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad + 2\,\frac{m(m-1)}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (G_Y(y))^2 f(y\,|\,x) f_{r:n}(x)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \left(m\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_Y(y) f(y\,|\,x) f_{r:n}(x)\,\mathrm{d}x\,\mathrm{d}y\right)^2.
\end{aligned}$$

Table 3
Some selected values for correlation coefficient of ξ_1 and ξ_2

n	r	m	α	$\rho(\xi_1, \xi_2)$
5	2	1	0.1	0.018
5	2	1	0.5	0.090
5	2	1	−0.5	−0.090
5	2	1	0.9	0.162
10	4	2	0.5	0.072
10	4	5	0.5	0.049
10	6	2	0.5	0.075
15	4	5	0.5	0.045

The covariance between ξ_1 and ξ_2

$$\begin{aligned}
&\mathrm{Cov}(\xi_1, \xi_2)\\
&= \sum_{i=1}^{m} [P\{X_{n+i} < X_{r:n}, Y_{n+i} < Y_{[r:n]}\} - P\{X_{n+i} < X_{r:n}\} P\{Y_{n+i} < Y_{[r:n]}\}]\\
&= m \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y) f(y \mid x) f_{r:n}(x) \,\mathrm{d}x \,\mathrm{d}y - \int_{-\infty}^{\infty} G_X(x) f_{r:n}(x) \,\mathrm{d}x \right.\\
&\quad \left. \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_Y(y) f(y \mid x) f_{r:n}(x) \,\mathrm{d}x \,\mathrm{d}y \right].
\end{aligned}$$

We give some numerical values for the correlation between ξ_1 and ξ_2 in the case of FGM distribution (see Table 3).

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