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On the General Class of Two-Sided Power Distribution

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Van Dorp and Kotz (2003) introduced a general class of two-sided power distribution. We provide several examples of this distribution by considering a non negative, non decreasing differentiable function g on $[0, 1]$ *satisfying* $g(1) - g(0) = 1$. *For the considered distributions the maximum likelihood estimation and moment estimation of parameters are obtained. Moments, moment generating functions, and relative entropy are also derived.*

Keywords Triangular distribution; Two-sided power distribution.

Mathematics Subject Classification Primary 60E05; Secondary 62E10.

1. Introduction

In recent years the triangular distribution and its extensions have aroused the interest of many statisticians. The papers by Johnson (1997) and Johnson and Kotz (1999) investigated the applications of this distribution in many areas as an alternative to the beta distribution. Van Dorp and Kotz (2002a) introduced an extension of the three-parameter triangular distribution which is called the standard two-sided power (TSP) distribution. The recent paper by Van Dorp and Kotz (2003) investigated a general form of a family of bounded two-sided continuous distributions using generating density. They also obtained the two-sided distributions by using integral convolution method.

The TSP distribution provides a reasonable alternative to the beta distribution in the view of the parameters.

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Let X be a random variable with probability density function (pdf) given by:

$$f(x \mid m, n) = \begin{cases} n\left(\dfrac{x}{m}\right)^{n-1}, & 0 < x \leq m \\ n\left(\dfrac{1-x}{1-m}\right)^{n-1}, & m \leq x < 1 \\ 0, & \text{elsewhere.} \end{cases} \tag{1.1}$$

X is said to be a standard TSP distribution TSP(m, n), $0 < m < 1, n > 1$. For the special values of parameters m and n, TSP(m, n) distribution reduces to uniform, triangular, and standard power distribution. Van Dorp and Kotz (2002a) derive the moment and maximum likelihood estimation (MLE) of the parameters and discuss possible applications of this distribution in Monte Carlo type uncertainty or risk analysis. Because of the flexibility, the TSP family may also be used as a rich family of prior distributions in Bayesian analysis. Van Dorp and Kotz (2002b) give a wide analysis of TSP distribution with applications in financial engineering. The recent work of Van Dorp and Kotz (2003) introduced another novel approach to TSP distributions. This class of distributions are generalized by using an appropriately selected continuous pdf supported on [0, 1].

In this study, we provide several examples of Van Dorp and Kotz generalizations of the TSP distributions which are flexible and have useful properties for applications. For certain generalized two-sided power distributions we evaluate moments, moment generating function, moment estimation, and MLE of parameters. Finally, we evaluate relative entropy of these distributions.

2. General Class of Two-Sided Power Distribution

Consider a non negative, non decreasing differentiable function $g(x)$, defined on [0, 1] with the property,

$$g(1) - g(0) = 1, \tag{2.1}$$

and denote $t(x)$ the derivative of $g(x)$. It is clear that $t(x)$ is a pdf on [0, 1].

Van Dorp and Kotz (2003) define a generalization of the two-sided power distribution (GTSP) with pdf

$$f(x \mid \theta) = \begin{cases} t\left(\dfrac{x}{\theta}\right), & 0 \leq x \leq \theta \\ t\left(\dfrac{1-x}{1-\theta}\right), & \theta \leq x \leq 1. \end{cases} \tag{2.2}$$

The cumulative distribution function of a GTSP (θ) is

$$F(x \mid \theta) = \begin{cases} \theta\left(g\left(\dfrac{x}{\theta}\right) - g(0)\right), & 0 \leq x \leq \theta \\ \theta - (1-\theta)\left\{g\left(\dfrac{1-x}{1-\theta}\right) - g(1)\right\}, & \theta \leq x \leq 1. \end{cases} \tag{2.3}$$

As a special case let us take $g(x) = x^n$ and substituting $g'(x)$ in Eq. (2.2), we obtain the classical standard two-sided power distribution TSP(θ, n). Note that $n = 1$ simplifies the above density function to a uniform density. Moreover, $n = 2$ gives the triangular distribution. A different type of distribution may be obtained by setting the kernel function $g(x)$.

1. Let $g(x)$ be

$$g(x) = \frac{e^{bx}}{e^b - 1}, \quad b > 0. \tag{2.4}$$

For fixed $b > 0$, (2.4) is monotonically increasing and convex function. The parameter b may be viewed as the shape parameter of convexity. For fixed x, (2.4) is monotonically decreasing for the parameter value b. Furthermore, it is observed that for fixed x in the interval (0, 1) in (2.4), as $b \to \infty$ the function $\frac{e^{bx}}{e^b-1} \sim (e^{x-1})^b$ and hence tends to 0. At $x = 1$, $\frac{e^{bx}}{e^b-1}$ tends to 1. These properties of (2.4) can be seen in Fig. 1.

From (2.4), it is clear that $t(x) = \frac{be^{bx}}{e^b-1}$. The pdf of GTSP distribution corresponding to (2.4) is given by

$$f(x \mid \theta, b) = \begin{cases} \dfrac{be^{\frac{bx}{\theta}}}{e^b - 1}, & 0 \le x \le \theta \\ \dfrac{be^{b\frac{(1-x)}{(1-\theta)}}}{e^b - 1}, & \theta \le x \le 1. \end{cases} \tag{2.5}$$

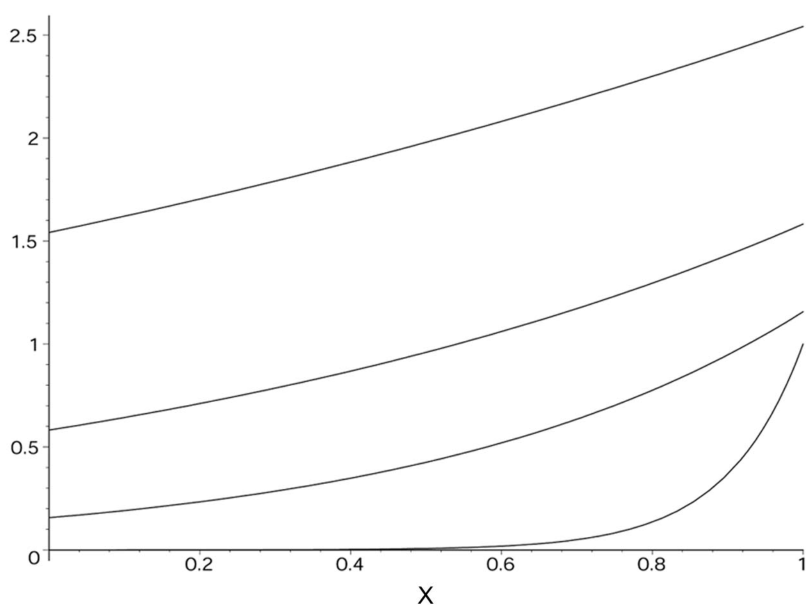

Figure 1. The function $g(x)$ with parameter $b = 0.5, 1, 2, 10$.

The cumulative distribution function of a GTSP (θ) is

$$F(x \mid \theta) = \begin{cases} \dfrac{\theta}{e^b - 1}\left(e^{\frac{bx}{\theta}} - 1\right), & 0 \le x \le \theta \\ \theta - (1-\theta)\left\{\dfrac{e^{b\left(\frac{1-x}{1-\theta}\right)}}{e^b - 1} - \dfrac{e^{\frac{b}{\theta}}}{e^b - 1}\right\}, & \theta \le x \le 1. \end{cases} \tag{2.6}$$

Figure 2 shows the behavior of GTSP density for different values of b and θ.

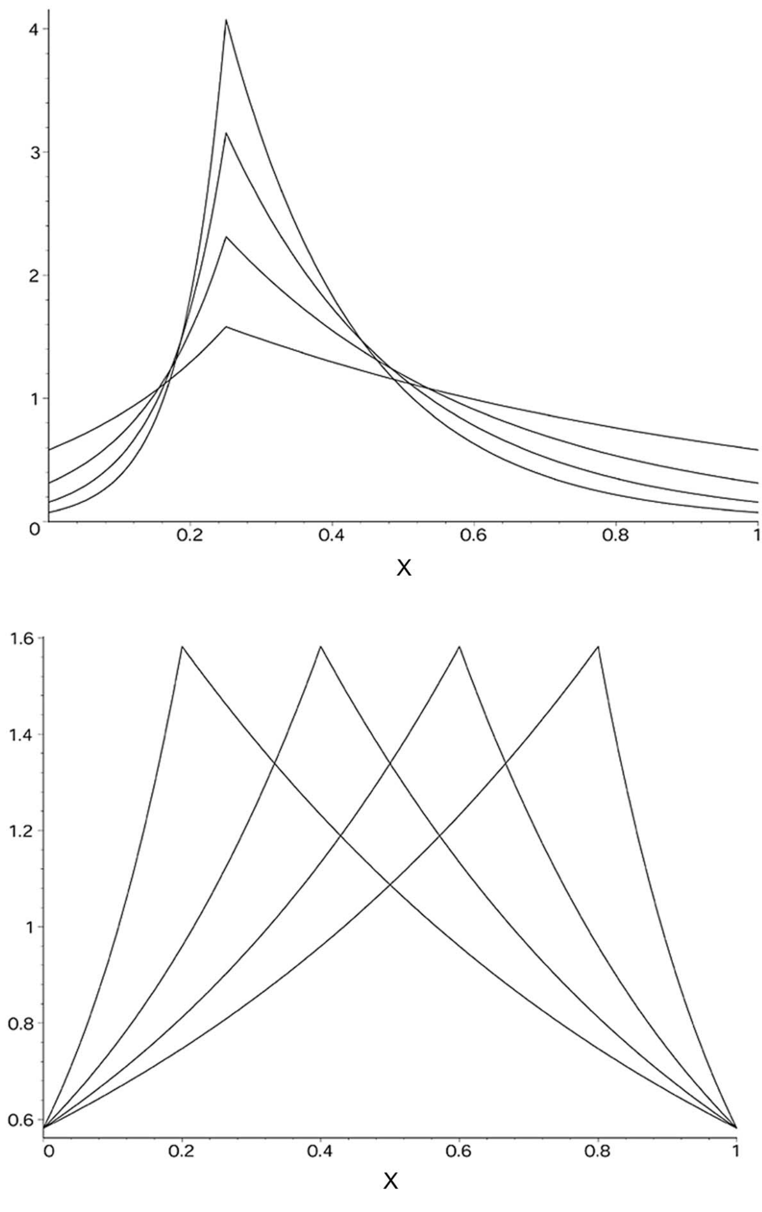

Figure 2. The GTSP density with $\theta = 0.25$ and $b = 1, 2, 3, 4$ and $\theta = 0.2, 0.4, 0.6, 0.8$ with $b = 1$.

The kth moment of (2.5) can be derived as

$$E(X^k) = \frac{\theta^{k+1}e^b}{e^b-1}\left\{\sum_{j=0}^{k}(-1)^j j!\binom{k}{j}\left(\frac{1}{b}\right)^j\right\}(-1)^{k+1}\frac{k!\theta^{k+1}}{b^k(e^b-1)} + \frac{(1-\theta)}{e^b-1}\sum_{j=0}^{k} j!\binom{k}{j}\{\theta^{k-j}e^b-1\}\left(\frac{1-\theta}{b}\right)^j. \quad (2.7)$$

It follows from (2.5) that the moment generating function is

$$M_X(t) = E(e^{tx}) = \frac{b}{e^b-1}\left(\frac{x(e^{(xt+b)}-1)}{xt+b} + \frac{(e^t - e^{(tx+b)})(1-x)}{(1-x)t-b}\right). \quad (2.8)$$

The expected value of the pdf (2.5) follows from (2.7) and is

$$E(X) = \frac{-1+b+e^b+e^b xb + xb + 2x - 2e^b x}{b(e^b-1)}. \quad (2.9)$$

For a fixed $b > 0$, the expected value (2.9) behaves like a linear function of x.

Using the densities given in Eq. (2.5) we obtain the likelihood function

$$L(x_1, x_2, \ldots, x_n; \theta) = \left(\frac{b}{e^b-1}\right)^n \exp\left(\frac{b}{\theta}\sum_{i=1}^{r} x_{(i)}\right)\exp\left(\frac{b}{1-\theta}\sum_{i=r+1}^{n}(1-x_{(i)})\right). \quad (2.10)$$

Then the MLE of θ is

$$\hat{\theta} = \sqrt{\frac{\sum_{i=1}^{r} x_{(i)}}{\sum_{i=r+1}^{n}(1-x_{(i)})}} + 1. \quad (2.11)$$

The derivative of the likelihood function with respect to b can be written as

$$\frac{\partial \ln L(x_1, x_2, \ldots, x_n; \theta)}{\partial b} = \frac{n}{b} - \frac{n}{1-\frac{1}{e^b}} + \frac{1}{\theta}\sum_{i=1}^{r} X_{(i)} + \frac{1}{1-\theta}\sum_{i=r+1}^{n}(1-X_{(i)}). \quad (2.12)$$

In order to obtain the MLE of the parameter b, we change $t = \frac{1}{e^b}$ and $b = \log\frac{1}{t}$ in (2.12). Then, using the approximation formula ($\log(1+t) \sim t$) in the last formula we obtain the MLE of b as

$$\hat{b} = -\ln\left|1 - \frac{n}{\frac{1}{\hat{\theta}}\sum_{i=1}^{r} X_{(i)} + \frac{1}{1-\hat{\theta}}\sum_{i=r+1}^{n}(1-X_{(i)})}\right|, \quad (2.13)$$

where $\hat{\theta}$ is the MLE of the θ.

2. Now, we consider a trigonometric function $g(x) = \sin\left(\frac{\pi x}{2}\right)$. Then $t(x) = \frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$. The pdf of GTSP distribution is given by

$$f(x \mid \theta) = \begin{cases} \frac{\pi}{2}\cos\left(\frac{\pi}{2}\frac{x}{\theta}\right), & 0 \le x \le \theta \\ \frac{\pi}{2}\cos\left(\frac{\pi}{2}\frac{1-x}{1-\theta}\right), & \theta \le x \le 1. \end{cases} \quad (2.14)$$

The cumulative distribution function is

$$F(x \mid \theta) = \begin{cases} \theta \sin\left(\frac{\pi}{2}\frac{x}{\theta}\right), & 0 \le x \le \theta \\ \theta - (1-\theta)\left\{\sin\left(\frac{\pi}{2}\left(\frac{1-x}{1-\theta}\right)\right) - 1\right\}, & \theta \le x \le 1. \end{cases} \tag{2.15}$$

Figure 3 shows the behavior of GTSP distribution (2.14) for different values of θ.

The moment generating function is given by

$$M_X(t) = -\frac{\pi\theta(-\pi e^{\theta t} + 2\theta t)}{4t^2\theta^2 + \pi^2} + \frac{\pi(2e^t t - 4e^t t\theta + 2e^t t\theta^2 + \pi e^{\theta t} - \pi\theta e^{\theta t})}{4t^2 - 8t^2\theta - 4t^2\theta^2 + \pi^2}. \tag{2.16}$$

The expected value is

$$E(X) = \frac{2 - 4\theta - \theta\pi}{\pi}, \tag{2.17}$$

which is a decreasing linear function for θ. The variance is

$$Var(X) = 4\frac{\pi - 3\theta\pi + 3\theta^2\pi - 3 + 10\theta - 10\theta^2}{\pi^2}. \tag{2.18}$$

Let $\theta = \frac{1}{2}$, then the expected value and variance are, respectively,

$$E(X) = -\frac{1}{2}, \quad Var(X) = \frac{\pi - 2}{\pi^2}. \tag{2.19}$$

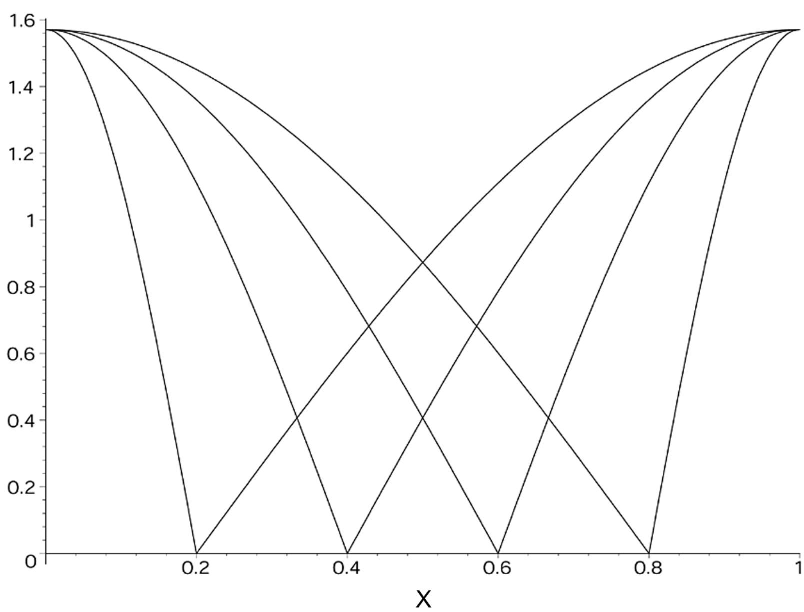

Figure 3. The GTSP density $f(x \mid \theta)$ with $\theta = 0.2, 0.4, 0.6, 0.8$.

Equating (2.17) the sample quantity $\bar{x}$, we find the moment estimation $\hat{\theta}$ of θ

$$\hat{\theta} = \frac{2 - \bar{x}\pi}{4 + \pi}.$$

3. Let $g(x)$ be

$$g(x) = \frac{1}{1 - \frac{x}{2}}. \tag{2.20}$$

Notice that $g(x)$ assumes the following power series expansion

$$g(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k, \quad x \in (-2, 2).$$

Then $t(x) = \frac{1}{2(1-\frac{1}{2}x)^2}$ and hence the pdf of GTSP distribution is given by

$$f(x \mid \theta) = \begin{cases} \dfrac{1}{2\left(1 - \frac{x}{2\theta}\right)^2}, & 0 \le x \le \theta \\ \dfrac{1}{2\left(1 - \frac{1}{2}\left(\frac{1-x}{1-\theta}\right)\right)^2}, & \theta \le x \le 1. \end{cases} \tag{2.21}$$

The cumulative distribution function of the latter pdf is

$$F(x \mid \theta) = \begin{cases} \dfrac{\theta x}{2\theta - x}, & 0 \le x \le \theta \\ \dfrac{x(2-\theta) - \theta}{x + 1 - 2\theta}, & \theta \le x \le 1. \end{cases} \tag{2.22}$$

Figure 4 depicts the behavior of the GTSP density for different values of θ.

Then, the moment generating function is given by

$$M_X(t) = 2e^{2\theta}t\theta^2[\Gamma(0, 2t\theta) - \Gamma(0, t\theta)] - \theta - e^t + \theta e^t$$
$$+ (1-\theta)^2 2te^{t(2\theta-1)}[\Gamma(0, t(\theta-1)) - \Gamma(0, 2t(\theta-1))],$$

where the incomplete Gamma function is defined by

$$\Gamma(a, b) = \int_b^{\infty} x^{a-1}e^{-x}\,dx, \quad \text{for } Re(a) > 0 \quad \text{and} \quad Re(x) > 0.$$

One may readily obtain the expected value

$$E(X) = 1 - 2\theta - \ln(2)(1 + 3\theta). \tag{2.23}$$

On using (2.21), one evaluates the variance

$$Var(X) = \ln(2)[8\theta^2 - 5\theta + 1] - \frac{1}{2}(11\theta^2 - 7\theta + 1) - [1 - 2\theta - \ln(2) + 3\theta\ln(2)]^2. \tag{2.24}$$

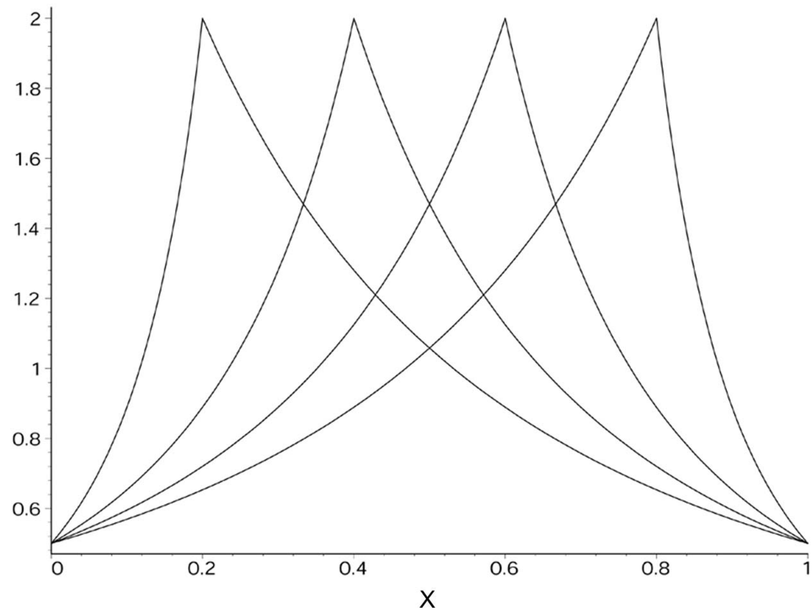

Figure 4. The GTSP density $f(x \mid \theta)$ with $\theta = 0.2, 0.4, 0.6, 0.8$.

There follows the moment estimation $\hat{\theta}$ of θ

$$\hat{\theta} = \frac{1 - \bar{x} - \ln(2)}{2 + 3\ln(2)}.$$

3. Relative Entropy

Information indices in many seemingly diverse problems can be explicated in a unified manner in terms of the discrimination information, also known as cross-entropy and relative entropy,

$$K(f : h \mid \theta) \equiv \int \log \frac{f(x \mid \theta)}{h(x)} dF(x \mid \theta), \tag{3.1}$$

where $f(x \mid \theta) = dF(x \mid \theta)$ is a pdf, absolutely continuous with respect to h. Relative entropy is used as a measure for comparing information content of distributions. The term discrimination, reflects the fact that $K(f : h \mid \theta) \geq 0$ and the equality holds if and only if $f(x \mid \theta) = h(x)$ almost everywhere, see e.g., Soofi and Retzer (2002). We compare the information contents of GTSP distributions on $[0, 1]$ with the information content of a uniform $[0, 1]$ distribution. After a change of variable we obtain that

$$\begin{aligned} K(t : h \mid \theta) &= \int_0^\theta \log\left(t\left(\frac{x}{\theta}\right)\right) t\left(\frac{x}{\theta}\right) dx + \int_\theta^1 \log\left(t\left(\frac{1-x}{1-\theta}\right)\right) t\left(\frac{1-x}{1-\theta}\right) dx \\ &= \int_0^1 t(x) \log(t(x)) dx. \end{aligned}$$

Let $g(x) = \frac{e^{bx}}{e^b - 1}$ and $t(x) = g'(x)$. Then the pdf of GTSP distribution is given in (2.5). It can be easily evaluated that the relative entropy of GTSP distributions with respect to a uniform [0, 1] distribution,

$$K(t : h \mid \theta) = \int_0^1 \frac{be^{bx}}{e^b - 1} \log \frac{be^{bx}}{e^b - 1}\, dx$$
$$= \frac{e^b \log\left(\frac{be^b}{e^b-1}\right) - e^b - \log\left(\frac{b}{e^b-1}\right) + 1}{e^b - 1}.$$

The relative entropy of GTSP distributions with respect to a uniform [0, 1] distribution increases very slowly and is asymptotically equal to $\log b$, since

$$\lim_{b\to\infty} \frac{\log\left(\frac{be^b}{e^b-1}\right)}{\log b} = 1 \quad \text{and} \quad \lim_{b\to\infty} \frac{\log\left(\frac{b}{e^b-1}\right)}{e^b} = 0.$$

Let $g(x) = \sin\left(\frac{\pi}{2}x\right)$ and $t(x) = g'(x)$. Then the pdf of GTSP distribution is obtained as in (2.14). The relative entropy of GTSP distribution (2.14) with respect to a uniform [0, 1] distribution is found as follows:

$$K(t : h \mid \theta) = \int_0^1 \frac{\pi}{2} \sin\left(\frac{\pi}{2}x\right) \log\left(\frac{\pi}{2} \sin\left(\frac{\pi}{2}x\right)\right) dx = \log(\pi/e).$$

Since relative entropy is a measure of inefficiency between two distributions, $K(f : h \mid \theta)$ shows a measure of inefficiency of the distribution $h(x)$ compared to the true distribution $f(x \mid \theta)$.

4. Conclusion

A characterization generating a wide variety of two-sided power distribution is discussed in this article. The GTSP distribution seems to be a useful and more flexible competitor to the beta distribution than the triangular distribution. It is expected that the initiation of the proposed characterization leading to several distributions into statistical application may bring fundamental innovations into the field of applied statistics and thus help practitioners in their succeeding studies.

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