Asymptotic distributions of the statistics based on order statistics, record values and invariant confidence intervals

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Abstract

In this work invariant confidence intervals for parametric class of distributions are considered. The finite sample and asymptotic properties of statistics based on invariant confidence intervals are investigated and their use in statistical inference is discussed.

Abstract

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INTRODUCTION

Let $X_1, X_2, ..., X_n$ be a sample from a distribution with distribution function (d.f) $F \in \Im$, where $\Im$ is some class of distribution functions. Suppose $f_1(u_1, u_2, ..., u_n)$ and $f_2(u_1, u_2, ..., u_n)$ are two Borel functions with the following property:

$$f_1(u_1, u_2, ..., u_n) \leq f_2(u_1, u_2, ..., u_n) \qquad (u_1, u_2, ..., u_n) \in R^n. \tag{1.1}$$

Let X_{n+1} be a new sample point obtained from F and be independent of $X_1, X_2, ..., X_n$. If

$$P\left\{X_{n+1} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n)\right\} = a = const \quad \text{for all} \quad F \in \Im,$$

then $(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n)$ is called an invariant confidence interval containing the main distributed mass for class of distributions $\Im$ with confidence level a .

It is well known that if $F \in \Im_c$, where $\Im_c$ denotes the class of all continuous distribution functions, then

$$P\left\{X_{n+1} \in (X_{(i)}, X_{(j)})\right\} = \frac{j-i}{n+1} \ ,$$

where $X_{(i)}$ denotes the i th order statistic of the sample $X_1, X_2, ..., X_n$, $1 \leq i < j \leq n$. Hence the random interval $(X_{(i)}, X_{(j)})$ is an invariant confidence interval for class of all continuous distribution functions. It is known that (see Bairamov, Petunin ,1991) if f_1 and f_2 are continuous, symmetric and different on every set with a nonzero lebesgue measure functions of n arguments, only the order statistics form invariant confidence intervals for $\Im_c$.

Properties of invariant confidence intervals for nonparametric class $\Im_c$ are utilized in many applications, a test statistic can be found and criteria can be established on the training samples for problems of classification of new observations (see Bairamov and Petunin ,1991; Bairamov, 1992). Similar applications can also be extended to generalized Bernoulli schemes in variation statistics (see Matveichuk and Petunin 1990; Matveichuk and Petunin 1991).

In this work we present the invariant confidence intervals for parametric class of distributions. The finite sample and asymptotic properties of statistics based on invariant confidence intervals are shown and their use in statistical inference is discussed.

1. INVARIANT CONFIDENCE INTERVALS FOR PARAMETRIC CLASS

Following part present invariant confidence intervals that contain main distributed mass of general set of parametric classes of distributions.

Assume that parametric family of distributions $\mathbf{P}$=$\{P_\theta, \theta \in \Theta\}$is given. Let $f_1(.)$ and $f_2(.)$ be n dimensional functions with the condition (1.1). Also assume that $X_1, X_2, ..., X_n, X_{n+1}$ is a random sample from the distribution $P_\theta \in \mathbf{P}$.

Definition 1.1. If the following is true,

$$P_\theta \{X_{n+1} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\} = \beta = const \quad \forall \theta \in \Theta, \tag{2.1}$$

then the random interval

$$(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n)) \equiv J(X_1, X_2, ..., X_n)$$

is called invariant confidence interval containing the main distributed mass for class $\mathbf{P}$ at confidence level β. •

It is obvious that if $X_{n+1}, X_{n+2}, ..., X_{n+m}$ is a new sample independent of $X_1, X_2, ..., X_n$, we can write for $\theta \in \Theta$

$$P_\theta \{X_{n+1}, ..., X_{n+m} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\} =$$

$$= \int ... \int [F_\theta(f_2(u_1, ..., u_n)) - F_\theta(f_1(u_1, ..., u_n))]^m \, dF_\theta(u_1)...dF_\theta(u_n) =$$

$$= E_\theta [F_\theta(f_2(X_1, X_2, ..., X_n)) - F_\theta(f_1(X_1, X_2, ..., X_n))]^m . \tag{2.2}$$

Denote

$$S_n(X_1, X_2, ..., X_n, \theta) = F_\theta(f_2(X_1, X_2, ..., X_n)) - F_\theta(f_1(X_1, X_2, ..., X_n)) \text{ and}$$
$$G_\theta(u) = P_\theta \{S_n(X_1, X_2, ..., X_n, \theta) \leq u\}.$$

Theorem 1.2. If the distribution of random variable (r.v.) $S_n(X_1, X_2, ..., X_n, \theta)$ is the same for all $\theta \in \Theta$ (i.e. the d.f of S_n is independent from θ), then $(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))$ interval is an invariant confidence interval for **P** family. •

Theorem 1.3. Let **P**=$\{P_\theta, \theta \in \Theta\}$ be a regular model , that is $\frac{dG_\theta(x)}{dx} = g_\theta(x)$ exist and is continuous, and $m(\theta) \equiv E_\theta(S^m)$ is differentiable for all θ as $m'(\theta) = \int u^m \frac{dg_\theta(u)}{d\theta} du, \theta \in \Theta$. Suppose that for $\forall \theta \in \Theta$

$$\begin{aligned} & P_\theta \{X_{n+1}, ..., X_{n+m} \in (f_1(X_1, X_2, ..., X_n), \\ f_2(X_1, X_2, ..., X_n))\} \quad \equiv \quad & d_m, m = N_1, N_2, ..., \sum N_i^{-1} = \infty \end{aligned}$$

(d_m is independent of θ, $m = N_1, N_2, ...$). Under such conditions the distribution of r.v.
$S_n(X_1, X_2, ..., X_n, \theta)$ is independent of θ. •

Theorem 1.2 and Theorem 1.3 pave a way of methods for constructing invariant confidence intervals. Development of these methods can be extended to families of distribution with location parameters. Let us assume that we have the family of distributions **P**= $\{F_\theta(x) = F(x - \theta)\ ,\ \theta \in \Theta\ ,\ F \text{ is known}\}$. If it is true that $X_1, X_2, ..., X_n$ has d.f. $F_\theta \in$**P**, then we can write

$$\overline{S_n}(X_1, X_2, ..., X_n, \theta) = F_\theta(f_2(X_1, X_2, ..., X_n)) - F_\theta(f_1(X_1, X_2, ..., X_n))$$

$$= F(f_2(X_1, X_2, ..., X_n) - \theta) - F(f_1(X_1, X_2, ..., X_n) - \theta)$$

Define $D^+ = \{(u_1, u_2, ..., u_n)\ ;\ u_1 \geq u_2 \geq ... \geq u_n\}$and let $a = (a_1, a_2, ..., a_n) \in R^n$, $b = (b_1, b_2, ..., b_n) \in R^n$, $a_{[1]} \geq a_{[2]} \geq ... \geq a_{[n]}$, $b_{[1]} \geq b_{[2]} \geq ... \geq b_{[n]}$ where $a_{[.]}$, $b_{[.]}$ designate order of magnitude. Given these definitions, under the conditions of

$$1. \sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}$$

$$2. \sum_{i=1}^{k} a_{[i]} \leq \sum_{i=1}^{k} b_{[i]}, k = 1, 2, ..., n - 1$$

it is said that vector a is majorant to vector b. This is expressed symbolically as $a \prec b$ (see Marshal and Olkin, 1979).

It is known that the necessary and sufficient condition for $a \prec b$ to hold true is the condition that

$$\sum_{i=1}^{n} a_i u_i \quad \leq \quad \sum_{i=1}^{n} b_i u_i, \text{for all } u = (u_1, u_2, ..., u_n) \in D^+.$$

(see Marshal and Olkin,1979, Chapter 4)

In order to utilize this theorem let $a = (\frac{1}{n}, \frac{1}{n}, ... \frac{1}{n})$, $b = (\frac{1}{n-1}, \frac{1}{n-1}, ... \frac{1}{n-1}, 0)$,and let $X_{[1]}, X_{[2]}, ..., X_{[n]}$ be the order statistics for the sample $X_1, X_2, ..., X_n$, $X_{[n-i+1]} = X_{(i)}$. Set

$$f_1(X_1, X_2, ..., X_n) = \frac{1}{n} \sum_{i=1}^{n} X_{[i]} \ , \ f_2(X_1, X_2, ..., X_n) = \frac{1}{n-1} \sum_{i=1}^{n-1} X_{[i]}.$$

Hence , it follows that

$$F_\theta \left(\frac{1}{n-1} \sum_{i=1}^{n-1} X_{[i]} \right) - F_\theta \left(\frac{1}{n} \sum_{i=1}^{n} X_{[i]} \right)$$

$$= \quad F \left(\frac{1}{n-1} \sum_{i=1}^{n-1} (X_{[i]} - \theta) \right) - F \left(\frac{1}{n} \sum_{i=1}^{n} (X_{[i]} - \theta) \right)$$

$$= F \left(\frac{1}{n-1} \sum_{i=1}^{n-1} (X_{(n-i+1)} - \theta) \right) - F \left(\frac{1}{n} \sum_{i=1}^{n} (X_{(n-i+1)} - \theta) \right).$$

It is obvious that , here the distribution of $X_{(n-i+1)} - \theta$ is independent of θ ; which means that distribution of r.v. $\overline{S_n}$ is the same for all elements of class $\mathbf{P}$. Now , we can consider another class of distributions; let us take the family of distributions $\mathbf{P}_1$ described as

$$\mathbf{P}_1 = \left\{ F_\theta(x) = F(\frac{x}{\theta}) \ , \ \theta \in \Theta \ , \ F(x) \text{ is known} \right\}.$$

For this class one can consider the r.v.

$$\tilde{S}_n(X_1, X_2, ..., X_n, \theta) \quad = \quad F_\theta \left(\sum_{i=1}^{n} b_i X_{[i]} \right) - F_\theta \left(\sum_{i=1}^{n} a_i X_{[i]} \right)$$

$$= \quad F \left(\frac{\sum_{i=1}^{n} b_i X_{[i]}}{\theta} \right) - F \left(\frac{\sum_{i=1}^{n} a_i X_{[i]}}{\theta} \right)$$

$$= F \left(\sum_{i=1}^{n} b_i \frac{X_{(n-i+1)}}{\theta} \right) - F \left(\sum_{i=1}^{n} a_i \frac{X_{(n-i+1)}}{\theta} \right).$$

Again since the distribution of $\frac{X_{(n-i+1)}}{\theta}$ is independent of θ , the distribution of $\tilde{S}_n \left(X_1, X_2, ..., X_n, \theta \right)$ is independent of θ. Obviously , it will be true and easy to show that for a two parameter family of distributions ;

$$\mathrm{P}_2 = \left\{ F_{\theta,\mu}(x) = F(\frac{x - \mu}{\theta}) \ , \ \theta \in \Theta \ , \ \mu \in \Theta_1 \ , \ F(x) \text{ is known} \right\}$$

the distribution of a similar random variable

$$S_n^*(X_1, X_2, ..., X_n, \theta, \mu) = F_{\theta,\mu}\left(\frac{1}{n-1}\sum_{i=1}^{n-1}(X_{[i]})\right) - F_{\theta,\mu}\left(\frac{1}{n}\sum_{i=1}^{n}(X_{[i]})\right),$$

is also independent from θ and μ.

1.1. The exponential distribution case

The exponential class of distributions occupies an important place in theory and application among the families of distributions. For this reason ; distribution free confidence intervals and statistics based on these intervals need to be discussed for this class.

Consider the class of distributions $\mathbf{P}_3 = \{P_\theta : P_\theta(x) = 1 - \exp(-\theta x)\ ,\ x \geq 0\ ,\ \theta > 0\}$. The parameter θ is a scale parameter for this family . Let $X_1, X_2, ..., X_n$ be a random sample with d.f. $P_\theta \in \mathbf{P}_3$, so

$$f_1(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} a_i X_{[i]} = \sum_{i=1}^{n} a_i X_{(n-i+1)}$$

$$f_2(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} b_i X_{[i]} = \sum_{i=1}^{n} b_i X_{(n-i+1)},$$

and $a \prec b$.

A distribution free confidence interval for $\mathbf{P}_3$ is conceived by the following theorems.

Theorem 1.4. For the class of exponential $\mathbf{P}_3$, it is true that

$$P_\theta\left\{X_{n+1} \in \left(\sum_{i=1}^{n} a_i X_{[i]}, \sum_{i=1}^{n} b_i X_{[i]}\right)\right\}$$

$$= \frac{n!}{\prod\limits_{j=1}^{n}\sum\limits_{i=1}^{j}(a_i+1)} - \frac{n!}{\prod\limits_{j=1}^{n}\sum\limits_{i=1}^{j}(b_i+1)}$$

$$= \alpha_1(a_1, a_2, ..., a_n; b_1, b_2, ..., b_n) = \beta_1\ ,\ \text{for } \forall\theta \in \Theta,$$

and $\left(\sum_{i=1}^{n} a_i X_{[i]}, \sum_{i=1}^{n} b_i X_{[i]}\right) = J(a, b, X_1, X_2, ..., X_n)$ is an invariant confidence interval for the class $\mathbf{P}_3$ at β_1 level. •

Corollary 1.5. Let , for example, $a = (0, 0, ..., 1)$ and $b = (1, 0, ..., 0), a \prec b$. Then $\sum_{i=1}^{n} a_i X_{[i]} = X_{[n]} = X_{(1)}$, $\sum_{i=1}^{n} b_i X_{[i]} = X_{[1]} = X_{(n)}$ and $\alpha_1(0, 0, ..., 1; 1, 0, ..., 0) = \frac{n-1}{n+1}$.•

Theorem 1.6. The probability that a new set of random sample values will fall in the interval $J(a, b, X_1, X_2, ..., X_n)$ is

$$P_\theta\{X_{n+1}, X_{n+2}, ..., X_{n+m} \in J(a, b, X_1, X_2, ..., X_n)\}$$

$$= \quad n!\sum_{k=0}^{m}\frac{(-1)^k\binom{m}{k}}{\prod\limits_{j=1}^{n}\sum\limits_{i=1}^{j}\{(m-k)\,a_i+kb_i+1\}}$$

$$\equiv \quad \beta_m(a_1,a_2,...,a_n;b_1,b_2,...,b_n)\equiv\beta_m.\bullet$$

Corollary 1.7. Let $a=(0,0,...,1)$ and $b=(1,0,...,0), a\prec b$. Then $\sum_{i=1}^{n}a_iX_{[i]}=X_{[n]}=X_{(1)}$, $\sum_{i=1}^{n}b_iX_{[i]}=X_{[1]}=X_{(n)}$. Then one can obtain from Theorem 1.6

$$P_\theta\left\{X_{n+1},X_{n+2},...,X_{n+m}\in(X_{(1)},X_n)\right\}=\frac{n!m!}{m+n}\sum_{k=0}^{m}\frac{(-1)^k}{(m-k)!(n-1+k)!}=$$

$$\frac{n!}{m+n}\sum_{k=0}^{m}\frac{(-1)^k m!k!(n-1)!}{(m-k)!k!(n-1+k)!(n-1)!}$$

$$= \quad \frac{n!}{(m+n)(n-1)!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{n-1+k}{k}^{-1}=$$

$$=\frac{n!}{(m+n)(n-1)!}.\frac{n-1}{n-1+m}=\frac{n(n-1)}{(m+n)(n-1+m)}. \tag{2.3}$$

Since $\mathbf{P}_3\subset\Im_c$ (2.3) may be obtained also from the following formula (see Bairamov and Petunin , 1991, Theorem 2)

$$P_F\left\{X_{n+1},X_{n+2},...,X_{n+m}\in(X_{(i)},X_{(j)})\right\}=\frac{n!(m+j-i-1)!}{(j-i-1)!(m+n)!}\qquad\forall F\in\Im_c,$$

taking $i=1$ and $j=n$. (Above in (2.3) we used the formula $\sum_{k=0}^{m}(-1)^k\binom{m}{k}\binom{a+k}{k}^{-1}=\frac{a}{a+m}$)$\bullet$

Theorem 1.4 and Theorem 1.5 above show that a new random sample of size m, X_{n+1}, X_{n+2}, ..., X_{n+m} will have observed values that fall in the distribution free random interval $J(a,b,X_1,X_2,...,X_n)$ and the probability of this random event is independent of exponential distribution parameter θ.

2. THE MAIN RESULT IN A TWO SAMPLE CASE

Let $\phi(u_1,u_2,...,u_n)$ be a real valued integrable n dimensional function. Consider a functional of it is defined as follows ;

$$H_F(\phi)=\int...\int\phi(u_1,u_2,...,u_n)dF(u_1)dF(u_2)....dF(u_n)\ ,\ F\in\mathbf{F}\ ,$$

where $\mathbf{F}$ is some class of distribution functions. The properties of functional $H_F(\phi)$ are

i) $H_F(1) = 1$
ii) $H_F(c_1\phi_1(.) + c_2\phi_2(.)) = c_1 H_F(\phi_1) + c_2 H_F(\phi_2)$.
Where $\phi_j(.)$ are distinct functions and c_j 's are real valued numbers.

Denote two random samples from two distributions $F(u)$ and $Q(u)$ as $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$, respectively. Let $f_1$and f_2 be the functions with properties expressed in (1.1). The probability of a random event

$$A_k = \{Y_k \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\} \ , \ k = 1, 2, ...m \ , \text{ is}$$

$$p \equiv P(A_k) = \int ... \int [Q(f_2(u_1, u_2, ..., u_n)) - Q(f_1(u_1, u_2, ..., u_n))] \, dF(u_1)dF(u_2)....dF(u_n),$$

which is independent of k, as seen. If we take definition of $H_F(\phi)$ above into consideration, the required probability for the each A_k is calculated by the following;

$$P(A_k) = p = H_F\left[Q(f_2(\bar{u})) - Q(f_1(\bar{u}))\right] \equiv H_F(Q_{f_1}^{f_2}(\bar{u})),$$

where $\bar{u} = (u_1, u_2, ..., u_n)$ and $Q(f_2(\bar{u})) - Q(f_1(\bar{u})) \equiv Q_{f_1}^{f_2}(\bar{u})$. Denoting ,

$$\xi_k = \begin{cases} 1, \text{ if random event } A_k \text{ is observed} \\ 0, \text{ if random event } A_k \text{ is not observed} \end{cases}$$

and defining a new random variable as $\nu_m = \xi_1 + \xi_2 + ... + \xi_m$, which can take values from the set $\{0, 1, 2, ..., m\}$, we can investigate the distributional properties of likeliness of observing new sampled values falling into a designated interval. Note that the r.v.'s $\xi_1, \xi_2, ..., \xi_m$ are dependent.

Theorem 2.1. For $k = 0, 1, 2, ..., m$ it is true that

$$P\{\nu_m = k\} = C_m^k H_F\left(\left[Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1 - Q_{f_1}^{f_2}(\bar{u})\right]^{m-k}\right) \ ,$$

where $C_m^k = \binom{m}{k} = \frac{m!}{k!(m-k)!}$, and mean and variance of ν_m are obtained as follows, respectively:

$$E(\nu_m) = mH_F(Q_{f_1}^{f_2}(\bar{u})),$$

$$\begin{aligned} var(\nu_m) &= m^2\left[H_F(Q_{f_1}^{f_2}(\bar{u}))^2 - \left(H_F(Q_{f_1}^{f_2}(\bar{u}))\right)^2\right] \\ &\quad -m\left[H_F(Q_{f_1}^{f_2}(\bar{u}))^2 - H_F(Q_{f_1}^{f_2}(\bar{u}))\right] .\bullet \end{aligned}$$

Lemma 2.2. The characteristic function for ν_m statistic is

$$\varphi_{\nu_m}(t) = H_F\left(1 + \left(e^{it} - 1\right) Q_{f_1}^{f_2}(u_1, u_2, ..., u_n)\right)^m \bullet .$$

Now let us define standardized ν_m as $\nu_m^* = \frac{\nu_m - E(\nu_m)}{\sqrt{var(\nu_m)}}$ with $E(\nu_m^*) = 0, var(\nu_m^*) = 1$. Denote

$$\begin{aligned} C(x) &= P\left\{Q_{f_1}^{f_2}(X_1, X_2, ..., X_n) \leq x\right\} \\ &= P\left\{Q(f_2(X_1, X_2, ..., X_n)) - Q(f_1(X_1, X_2, ..., X_n)\ ) \leq x\right\} \end{aligned}$$

.

Theorem 2.3. Let f_1 and f_2 be continuous functions, F and Q are continuous d.f.'s. Then it is true that

$$\lim_{m\to\infty} \sup_{0\leq x\leq 1} \left| P\left\{\frac{\nu_m}{m} \leq x\right\} - C(x)\right| = 0.\bullet$$

The following results follow from Theorem 2.3.

Corollary 2.4. Denote $a = A(F,Q) = H_F(Q_{f_1}^{f_2}(\bar{u}))^2$ and $b = B(F,Q) = H_F(Q_{f_1}^{f_2}(\bar{u}))$. Then under conditions of Theorem 2.3 it is true

$$\lim_{m\to\infty} \sup_{-\frac{b}{a}\leq x\leq \frac{1-b}{a}} \left| P\left\{\nu_m^\star \leq x\right\} - C_1(x)\right| = 0,$$

where $C_1(x) = C(ax+b)$ and $\nu_m^\star = \frac{\nu_m - E\nu_m}{\sqrt{m^2(a-b^2)-m(a-b)}}$.

Corollary 2.5. Let $(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))$ be the invariant confidence interval for some class of distributions $\Im$ with confidence level α_1, i.e.

$$P_F\left\{X_{n+1} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\right\} = \alpha_1 \text{ for any } F \in \Im.$$

Denote $\alpha_2 = P_F\left\{X_{n+1}, X_{n+2} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\right\}$, where $X_1, X_2, ..., X_n,\ X_{n+1}, X_{n+2}$ is the random sample from distribution with d.f. $F \in \Im$. Let $F = Q$ and $F \in \Im$, $X = (X_1, X_2, ..., X_n)$. Then

$$\lim_{m\to\infty} \sup_{x} \left| P\left\{\frac{\nu_m - m\alpha_1}{\sqrt{m^2(\alpha_2 - \alpha_1^2) - m(\alpha_2 - \alpha_1)}} \leq x\right\} - C_2(x)\right| = 0,$$

where

$$C_2(x) = \left\{ \begin{array}{lll} 0, & if & x \leq -\frac{\alpha_1}{\sqrt{\alpha_2 - \alpha_1^2}} \\ P\left\{F(f_2(X)) - F(f_1(X)) \leq \sqrt{\alpha_2 - \alpha_1^2}x + \alpha_1\right\}, & if & x \in \left(-\frac{\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}}, \frac{1-\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}}\right) \\ 1, & if & x \geq \frac{1-\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}} \end{array} \right. .\bullet$$

Corollary 2.6. Let **P**=$\Im_c$, where $\Im_c$ is the family of all continuous distributions. Let

$f_1(X_1,X_2,...,X_n)=X_{(i)}, \quad f_2(X_1,X_2,...,X_n)=X_{(j)}, 1\le i<j\le n$. Given all these we can write and show that (see Bairamov, Petunin,1991)

$$H_F\left(F_{u_{(i)}}^{u_{(j)}}\left(u_1,u_{2,...,}u_n\right)\right)=P\left\{X_{n+1}\in\left(X_{(i)},X_{(j)}\right)\right\}=\frac{j-i}{n+1}\equiv\alpha_{i,j} \text{ and}$$

$$H_F\left[\left(F_{u_{(i)}}^{u_{(j)}}\left(u_1,u_{2,...,}u_n\right)\right)^m\right]=P\left\{X_{n+1},X_{n+2},...,X_{n+m}\in\left(X_{(i)},X_{(j)}\right)\right\}$$
$$=\frac{n!(m+j-i-1)!}{(j-i-1)!(m+n)!}\equiv\alpha_{ij}^{(m)}.$$

If $i=1$ and $j=n$ then $\alpha_{1,n}=\frac{n-1}{n+1}$, $\alpha_{1,n}^{(2)}=\frac{(n-1)n}{(n+1)(n+2)}$.

Corollary 2.7. Let $X_1,X_2,...,X_n$ be a sample with d.f $F\in$**P**$=\Im_c$, where $\Im_c$ is the family of all continuous distributions. Let $f_1(X_1,X_2,...,X_n)=X_{(i)}$, , $f_2(X_1,X_2,...,X_n)=X_{(j)}, 1\le i<j\le n$. In this case, $C(x)$ in Theorem 2.3 has the form $C(x)=P_F\left\{Q(X_{(j)})-Q(X_{(i)})\le x\right\}$. If $F=Q$ then $C(x)=P\left\{F(X_{(j)})-F(X_{(i)})\le x\right\}=P\left\{W_{ij}\le x\right\}$, where W_{ij} has beta distribution with parameter $(j-i,n-j+i+1)$ (see David, 1970). This result conforms with Theorem 3.3. in [4] . It is not difficult to see that a and b in Corollary 2.5 have the magnitudes $a=\sqrt{\frac{(j-i)(j-i+1)}{(n+1)(n+2)}-\frac{(j-i)^2}{(n+1)^2}}$, $b=\frac{j-i}{n+1}$.

3. AN APPLICATION IN HYPOTHESIS TESTING

Let **P**$=\{P_\theta,\theta\in\Theta\}$ parametric family of distributions which is given. Also let the random sample

$X_1,X_2,...,X_n,\ X_{n+1},X_{n+2},...,X_{n+m}$ be drawn from distribution with d.f. F. In order to test if this distribution belongs to **P** family or not; composite hypothesis $H_0: F\in$**P** is set against the alternative composite hypothesis $H_1: F\notin$**P** .

To carry out the test, the following procedure is followed. Divide the random sample $X_1,X_2,...,X_n,\ X_{n+1},X_{n+2},...,X_{n+m}$ into two parts as $X_1,X_2,...,X_n$ and $X_{n+1},X_{n+2},...,X_{n+m}$.

Take $f_1(\overline{u})\le f_2(\overline{u})$, $\overline{u}\in R^n$ such that

$$P_\theta\left\{P_\theta(f_2(X_1,X_2,...,X_n)-P_\theta(f_1(X_1,X_2,...,X_n))\le x\right\}=D(x)\quad\forall\theta\in\Theta.$$

Then by (2.1)

$$P_\theta\left\{X_{n+1}\in(f_1(X_1,X_2,...,X_n),f_2(X_1,X_2,...,X_n))\right\}\equiv\alpha_1=const\quad\forall\theta\in\Theta,$$

$$P_\theta\left\{X_{n+1},X_{n+2}\in(f_1(X_1,X_2,...,X_n),f_2(X_1,X_2,...,X_n))\right\}$$
$$\equiv\quad\alpha_2=const\ \forall\theta\in\Theta..$$

Let

$$\nu_{m=}\nu_m(X_1, X_2, ..., X_n, X_{n+1}, X_{n+2}, ..., X_{n+m})$$

statistic be expressed as the number of observations from $X_{n+1}, X_{n+2}, ..., X_{n+m}$ that fall into

$$(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))$$

interval. By the result presented in section 2 above $E\nu_m = m\alpha_1$, $var(\nu_m) = m^2(\alpha_2 - \alpha_1^2) - m(\alpha_2 - \alpha_1)$ and by Corollary 2.5 the following is obtained if H_0 is true:

$$\lim_{m\to\infty} \sup_x \left| P\left\{ \frac{\nu_m - m\alpha_1}{\sqrt{m^2(\alpha_2 - \alpha_1^2) - m(\alpha_2 - \alpha_1)}} \le x \right\} - D_1(x) \right| = 0,$$

where

$$D_1(x) = \begin{cases} 0 & if \quad x \le -\frac{\alpha_1}{\sqrt{\alpha_2 - \alpha_1^2}} \\ D(x\sqrt{\alpha_2 - \alpha_1^2} + \alpha_1) & if \quad x \in \left(-\frac{\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}}, \frac{1-\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}} \right) \\ 1 & if \quad x \ge \frac{1-\alpha_1}{\sqrt{\alpha_2-\alpha_1^2}} \end{cases}.$$

Accordingly, the critical region for the test, that we will call W_α^{m+n} is defined as

$$W_\alpha^{m+n} = \left\{ \left| \frac{\nu_m - m\alpha_1}{\sqrt{m^2(\alpha_2 - \alpha_1^2) - m(\alpha_2 - \alpha_1)}} \right| > x_\alpha \right\}$$

and when n is kept fixed it occurs in the limit that

$$\lim_{m\to\infty} P\left\{ (X_1, X_2, ..., X_n, X_{n+1}, ..., X_{n+m}) \in W_\alpha^{m+n} \mid H_0 \right\} = 1-(D_1(x_\alpha)-D_1(-x_\alpha)) = 1-\alpha. \tag{4.1}$$

Let us investigate the power of W_α^{m+n} in point $Q \notin \mathbf{P}$**,** when m and n tends to ∞. By using Theorem 2.3 one can write

$$P_Q\left\{ \left| \frac{\nu_m - m\alpha_1}{\sqrt{m^2(\alpha_2 - \alpha_1^2) - m(\alpha_2 - \alpha_1)}} \right| > x_\alpha \right\} = 1-P_Q\left\{ \alpha_1 - x_\alpha\sqrt{(\alpha_2 - \alpha_1^2) - \frac{\alpha_2 - \alpha_1}{m}} \le \right.$$

$$\left. \le \frac{\nu_m}{m} \le \alpha_1 + x_\alpha\sqrt{(\alpha_2 - \alpha_1^2) - \frac{\alpha_2 - \alpha_1}{m}} \right\} \xrightarrow{w}$$

$$1 - \left[P_Q\left\{ Q(f_2(X)) - Q(f_1(X)) \le \alpha_1 + x_\alpha\sqrt{\alpha_2 - \alpha_1^2} \right\} - \right.$$

$$\left. -P_Q\left\{ Q(f_2(X)) - Q(f_1(X)) \le \alpha_1 - x_\alpha\sqrt{\alpha_2 - \alpha_1^2} \right\} \right]. \tag{4.2}$$

It is seen from (4.2) that if $\alpha_2 - \alpha_1^2$ tends to zero as n tends to infinity then W_α^{m+n} will be consistent for testing H_0 against H_1.

Example 3.1. Assume we want to test the hypothesis H_0 asserting that the sample $X_1, X_2, ..., X_n,\ X_{n+1}, X_{n+2}, ..., X_{n+m}$ has d.f $P_\theta(x) = 1 - e^{-\frac{x}{\theta}}, x \geq 0$, where $\theta > 0$ is unknown parameter, against H_1 asserting that X is not exponential. Let $f_1(X_1, X_2, ..., X_n) = -\infty,\ f_2(X_1, X_2, ..., X_n) = \sum_{i=1}^{n} X_{(i)} = \sum_{i=1}^{n} X_i$. From results of Section 1 it follows that $(-\infty, \sum_{i=1}^{n} X_i) \equiv J$ is an invariant confidence interval for the class $\mathbf{P}$=$\{P_\theta : P_\theta(x) = 1 - e^{-\frac{x}{\theta}}, x \geq 0, \theta > 0.\}$ Consider $\xi_i = \begin{cases} 1, & if \quad X_{n+i} \in J \\ 0, & if \quad X_{n+i} \notin J \end{cases}, i = 1, 2, ..., m$ and $\nu_m = \sum_{i=1}^{m} \xi_i$. The values of $P_\theta \{X_{n+1} \in J\} = \alpha_1$ and $P_\theta \{X_{n+1}, X_{n+2} \in J\} = \alpha_2$ are computed using Theorem 1.4 and Theorem 1.6 It is clear that if H_0 is true then

$$D(x) = P_\theta \left\{ P_\theta(\sum_{i=1}^{n} X_i) \leq x \right\} = P_\theta \left\{ 1 - \exp\left\{ -\sum_{i=1}^{n} \frac{X_i}{\theta} \right\} \leq x \right\}$$

$$= P \left\{ 1 - \exp\left\{ -\sum_{i=1}^{n} Z_i \right\} \leq x \right\} = P \left\{ \sum_{i=1}^{n} Z_i \leq \ln \frac{1}{1-x} \right\} = \Gamma_{n,1}(\ln \frac{1}{1-x}),$$

where $Z_i \overset{d}{=} \frac{X_i}{\theta}$ and $P\{Z_i \leq x\} = 1 - e^{-x}, x \geq 0$; $\Gamma_{n,1}(x)$ is d.f of gamma distribution with parameters $(n, 1)$.

4. PROOFS OF THEOREMS

Proof of Theorem 1.2. Consider the equality (2.2). Denote $P_\theta \{S_n(X_1, X_2, ..., X_n, \theta) \leq u\}$ $= G(u)$. Since $G(u)$ is independent of θ , the following equality holds true.

$$E_\theta \left(S_n(X_1, X_2, ..., X_n, \theta)\right)^m = \int u^m dG(u) = a_m \ , m = 1, 2, ... \ \bullet$$

Proof of Theorem 2.3. Under the stated conditions in the theorem , we can write

$$P_\theta \{X_{n+1}, ..., X_{n+m} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\}$$

$$= \int_0^1 u^m dG_\theta(u) = \int_0^1 u^m g_\theta(u) du = d_m, m = N_1, N_2, ... \quad \sum_i N_i^{-1} = \infty,$$

since the model is regular, we have $\int u^m \frac{dg_\theta(u)}{d\theta} du = 0$, $m = N_1, N_2, ...$. According to Müntz-Szass theorem (see Akhiezer, 1961) the set of functions $u^{N_1}, u^{N_2}, ...$ with $\sum_i N_i^{-1} = \infty$ is closed in the space of all continuous functions $C_{[0,1]}$. Then we obtain $\frac{dg_\theta(u)}{d\theta} = 0$ and this shows that $\frac{dG_\theta(u)}{du} = g_\theta(u)$ is independent of θ. $\bullet$

Proof of Theorem 1.4. We can write

$$P_\theta\left\{X_{n+1} \in \left(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n)\right)\right\}$$
$$= P\left\{X_1, X_2, ..., X_n, X_{n+1} \in G :\right.$$
$$\left. G = \left\{(x_1, ..., x_n, x_{n+1}) : x_{n+1} \in (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n)), 0 < x_n < ... < x_1 < \infty\right\}\right\}$$

$$= \int_0^\infty \int_0^{x_1} ... \int_0^{x_{n-1}} \int_{f_1(x_1,...,x_n)}^{f_2(x_1,...,x_n)} n! dF(x_{n+1}) dF(x_n) ... dF(x_1)$$

$$= n! \int_0^\infty \int_0^{x_1} ... \int_0^{x_{n-1}} \left[F(f_2(x_1, ..., x_n)) - F(f_1(x_1, ..., x_n))\right] dF(x_n) ... dF(x_1)$$

$$n! \int_0^\infty \int_0^{x_1} ... \int_0^{x_{n-1}} \left[\left(1 - \exp(-\theta \sum_{i=1}^n b_i x_i)\right) - \left(1 - \exp(-\theta \sum_{i=1}^n b_i x_i)\right)\right] \times$$
$$\times \theta^n \exp(-\theta \sum_{i=1}^n x_i) dx_1 ... dx_n$$

$$\frac{n!}{\prod_{j=1}^n \sum_{i=1}^j (a_i + 1)} - \frac{n!}{\prod_{j=1}^n \sum_{i=1}^j (b_i + 1)}. \quad \bullet$$

Proof of Theorem 1.6. We can express in detail that

$$P_\theta\left\{X_{n+1}, X_{n+2}, ..., X_{n+m} \in J(a, b, X_1, X_2, ..., X_n)\right\}$$

$$= P_\theta\left\{(X_1, ..., X_n, X_{n+1}, ..., X_{n+m}) \in G = \left\{(x_1, ..., x_n, x_{n+1}, ..., x_{n+m}) :\right.\right.$$
$$x_{n+1} \in (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n)), x_{n+2} \in (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n))$$
$$\left.\left., ..., x_{n+m} \in (f_1(x_1, ..., x_n), f_2(x_1, ..., x_n)), 0 < x_n < x_{n-1} < ... < x_2 < x_1\right\}\right\}$$

$$= n! \int_0^\infty \int_0^{x_1} ... \int_0^{x_{n-1}} \int_{f_1(x_1,...,x_n)}^{f_2(x_1,...,x_n)} ... \int_{f_1(x_1,...,x_n)}^{f_2(x_1,...,x_n)} dF(x_{n+m}) ... dF(x_{n+1}) dF(x_n) ... dF(x_1)$$

$$= n! \int_0^\infty \int_0^{x_1} ... \int_0^{x_{n-1}} \left[F(f_2(x_1, ..., x_n)) - F(f_1(x_1, ..., x_n))\right]^m dF(x_n) ... dF(x_1)$$

$$= n! \sum_{k=0}^m \frac{(-1)^k \binom{m}{k}}{\prod_{j=1}^n \sum_{i=1}^j \left\{(m-k) a_i + k b_i + 1\right\}}. \quad \bullet$$

Proof of Theorem 2.1. Let $A_{ik} = \{Y_{ik} \in (f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))\}$.
The probability that k of $Y_1, Y_2..., Y_m$ values fall in the interval $(f_1(X_1, X_2, ..., X_n), f_2(X_1, X_2, ..., X_n))$ is

$$P\{\nu_m = k\} = \sum_{i_1, i_2, .., i_m} P(A_{i1}, A_{i2}, ..., A_{ik}, \overline{A_{i_{k+1}}}, \overline{A_{i_{k+2}}}, ..., \overline{A_{i_m}}). \qquad (5.1)$$

Where $\overline{A}$ denotes the complement of A. Let us say that

$$P^{(k)}_{i_1,i_2,..,i_m} = P(A_{i1}, A_{i2}, ..., A_{ik}, \overline{A_{i_{k+1}}}, \overline{A_{i_{k+2}}}, ..., \overline{A_{i_m}}).$$

In this case, it is true that

$$P^{(k)}_{i_1,i_2,..,i_m} = \int \underset{\mathbf{A}}{...} \int dF(y_{i_1}, y_{i_2}, ..., y_{i_m}, x_1, x_2, ..., x_n) \text{ and}$$

$$\mathbf{A} = \{(y_{i_1}, y_{i_2}, ..., y_{i_m}, x_1, x_2, ..., x_n) : -\infty < x_i < \infty \ , \ i = 1, 2, ..., n \ ;$$
$$f_1(x_1, x_2, ..., x_n) < y_{i_p} < f_2(x_1, x_2, ..., x_n), p = 1, 2, ..., k;$$
$$y_{i_j} \notin (f_1(x_1, x_2, ..., x_n), f_2(x_1, x_2, ..., x_n)), \ j = k+1, k+2, ..., m\Big\}.$$

$Y_1, Y_2..., Y_m$ and $X_1, X_2..., X_n$ have the following joint distribution function by independence;

$$F(y_{i_1}, y_{i_2}, ..., y_{i_m}, x_1, x_2, ..., x_n) = Q(y_1)Q(y_2)..Q(y_m)F(x_1)F(x_2)...F(x_n).$$

So we can write

$$P^{(k)}_{i_1,i_2,..,i_m} = \int \underset{\mathrm{A}}{...} \int [Q(f_2(u_1, u_2, ..., u_n)) - Q(f_1(u_1, u_2, ..., u_n))]^k$$

$$\text{x} [1 - Q(f_2(u_1, u_2, ..., u_n)) + Q(f_1(u_1, u_2, ..., u_n)]^{m-k} dF(u_1)...dF(u_n)$$
$$= H_F\left(\left[Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^k \left[1 - Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^{m-k}\right).$$

As sown here: $P^{(k)}_{i_1,i_2,..,i_m}$ probabilities in summation (5.1) are independent of $i_1, i_2, ..., i_m$. The low order moments of interest for further uses are expressed as follows

$$E(\nu_m) = \sum_{k=0}^{m} kP\{\nu_m = k\}$$

$$= \sum_{k=0}^{m} kC^k_m \int_{-\infty}^{\infty} \cdots \int \left[Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^k \left[1 - Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^{m-k} dF(u_1)...dF(u_n)$$

$$= \int_{-\infty}^{\infty} \cdots \int \left(\sum_{k=0}^{m} kC^k_m \left[Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^k \left[1 - Q^{f_2}_{f_1}(u_1, u_2, ..., u_n)\right]^{m-k}\right) dF(u_1)...dF(u_n).$$

Now, we need to find the summation

$$\sum_{k=0}^{m} kC_m^k \left[Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k}.$$

In detailed expression we find that

$$\sum_{k=0}^{m} kC_m^k \left[Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k} = m\sum_{k=1}^{m} C_{m-1}^{k-1} \left[Q_{f_1}^{f_2}(\bar{u})\right]^{k-1} \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k} \left[Q_{f_1}^{f_2}(\bar{u})\right]$$

$$= mQ_{f_1}^{f_2}(\bar{u}) \sum_{i=0}^{m-1} C_{m-1}^{i} \left[Q_{f_1}^{f_2}(\bar{u})\right]^{i} \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{(m-1)-i} = mQ_{f_1}^{f_2}(\bar{u}).$$

Finally, we obtain

$$E(\nu_m) = m\int_{-\infty}^{\infty}\cdots\int Q_{f_1}^{f_2}(u_1,u_2,...,u_n)dF(u_1)...dF(u_n) = mH_F\left[Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right]$$

It is also found that

$$E(\nu_m^2) = m^2 H_F\left[Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right]^2 - mH_F\left[Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right]^2 + mH_F\left[Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right].\bullet$$

Proof of Lemma 2.2. By definition,

$$\varphi_{\nu_m}(t) = E\left(\exp(it\nu_m)\right) = \sum_{k=0}^{m} \exp(itk)P(\nu_m = k)$$

$$= \sum_{k=0}^{m} \exp(itk) C_m^k H_F \left[Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k}$$

$$= \int_{-\infty}^{\infty}\cdots\int \sum_{k=0}^{m} \exp(itk) C_m^k \left[Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right]^k \left[1-Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\right]^{m-k} dF(u_1)...dF(u_n)$$

In order to carry out the calculation, first summation term is found as

$$\sum_{k=0}^{m} \exp(itk) C_m^k \left[Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k} = \sum_{k=0}^{m} C_m^k \left[\exp(it) Q_{f_1}^{f_2}(\bar{u})\right]^k \left[1-Q_{f_1}^{f_2}(\bar{u})\right]^{m-k}$$

$$= \left(\exp(it)Q_{f_1}^{f_2}(\bar{u}) + \left(1-Q_{f_1}^{f_2}(\bar{u})\right)\right)^m = \left(1-Q_{f_1}^{f_2}(\bar{u})\left(1-\exp(it)\right)\right)^m.$$

So we can write

$$\varphi_{\nu_m}(t) = \int_{-\infty}^{\infty}\cdots\int \left(1-Q_{f_1}^{f_2}(u_1,u_2,...,u_n)\left(1-\exp(it)\right)\right)^m dF(u_1)...dF(u_n)$$

$$= H_F \left(1 + (\exp(it) - 1)\, Q_{f_1}^{f_2}(u_1, u_2, ..., u_n)\right)^m . \bullet$$

Proof of Theorem 2.3. By using Lemma 3.1 the characteristic function of $\frac{\nu_m}{m}$ can be written as

$$\varphi_{\frac{\nu_m}{m}}(t) = E(e^{i\frac{t}{m}\nu_m}) = H_F\left(1 + \left(\exp(i\frac{t}{m}) - 1\right) Q_{f_1}^{f_2}(\bar{u})\right)^m \tag{5.2}$$

Denote $\Psi_m(t) = \left(1 + \left(\exp(i\frac{t}{m}) - 1\right) Q_{f_1}^{f_2}(\bar{u})\right)^m$. Using Taylor expansion $e^x = 1 + x + o(x)$ and $\ln(1 + x) = x + o(x)$ one can write

$$\ln \Psi_m(t) = m \ln\left(1 + \left(\exp(i\frac{t}{m}) - 1\right) Q_{f_1}^{f_2}(\bar{u})\right) = m \ln\left(1 + \left(\frac{it}{m} + o(\frac{it}{m})\right) Q_{f_1}^{f_2}(\bar{u})\right)$$

$$= m \ln\left(1 + \left(\frac{it}{m} Q_{f_1}^{f_2}(\bar{u}) + o(\frac{t}{m})\right)\right) = m\left(\frac{it}{m} Q_{f_1}^{f_2}(\bar{u}) + o(\frac{t}{m})\right) = itQ_{f_1}^{f_2}(\bar{u}) + O(\frac{1}{m})$$

Then

$$\Psi_m(t) = \exp\left(itQ_{f_1}^{f_2}(\bar{u})\right) + O(\frac{1}{m})$$

It follows from (5.2) that

$$\varphi_{\frac{\nu_m}{m}}(t) = H_F(\Psi_m(t)) = H_F\left(\exp\left(itQ_{f_1}^{f_2}(\bar{u})\right)\right) + O(\frac{1}{m}). \tag{5.3}$$

Letting to limit in (5.3) we obtain

$$\lim_{m\to\infty} \varphi_{\frac{\nu_m}{m}}(t) = H_F\left(\exp\left(itQ_{f_1}^{f_2}(\bar{u})\right)\right) \equiv \Psi(t). \tag{5.4}$$

It easy to see that $\Psi(t)$ is continuous at the point $t = 0$. In fact one has

$$\exp\left(itQ_{f_1}^{f_2}(\bar{u})\right) = 1 + itQ_{f_1}^{f_2}(\bar{u}) + \frac{i^2t^2\left(Q_{f_1}^{f_2}(\bar{u})\right)^2}{2!} + o(t^2)$$

and $\Psi(t) = H_F\left(\exp\left(itQ_{f_1}^{f_2}(\bar{u})\right)\right) \to 1 = \Psi(0)$ if $t \to 0$.

Let $F_m^*(x)$ be the d.f. of statistic ν_m, where $x = \frac{k}{m}$, $k = 0, 1, 2, ..., m$. By using Levy-Crammer theorem for characteristic functions (see Petrov,1975,Theorem 10,P.15) one can obtain that $F_m^*(x) \to F^*(x), x \in [0, 1]$ and F^* has a characteristic function

$$\Psi(t) = \int_0^1 e^{itx} dF^*(x). \tag{5.5}$$

On other hand; from (5.4) we have

$$\Psi(t) = H_F\left(\exp\left(itQ_{f_1}^{f_2}(\bar{u})\right)\right) = \int_{-\infty}^{\infty}\cdots\int \exp\left(itQ_{f_1}^{f_2}(u_1, u_2, ..., u_n)\right) dF(u_1)...dF(u_2)$$

$$= E\left[\exp\left(itQ_{f_1}^{f_2}(X_1, X_2, ..., X_n)\right)\right] = \int_0^1 e^{itx} dP\left\{Q_{f_1}^{f_2}(X_1, X_2, ..., X_n) \le x\right\} \tag{5.6}$$

Therefore from (5.5) and (5.6) we have $F^*(x) = P\left\{Q_{f_1}^{f_2}(X_1, X_2, ..., X_n) \le x\right\} =$

$$F^*(x) = P\left\{Q_{f_1}^{f_2}(X_1, X_2, ..., X_n) \le x\right\} = P\left\{Q\left(f_2(X_1, ..., X_n)\right) - Q\left(f_1(X_1, ..., X_n)\right) \le x\right\}. \bullet$$

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