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Dependence structure and symmetry of Huang-Kotz FGM distributions and their extensions

Ismihan Bairamov[1], **Samuel Kotz**[2]

[1] Department of Mathematics, Izmir University of Economics, 35330, Balcova, Izmir, Turkey (e-mail: bayramov@science.ankara.edu.tr)
[2] Department of Engineering Management, The George Washington University, Washington, D.C., 200052, USA (e-mail: kotz@seas.gwu.edu)

Abstract. An extension of FGM class of bivariate distributions with given marginals is presented. For Huang-Kotz FGM distributions some theorems characterizing symmetry and conditions for independence are obtained. The new family of distributions allows us to achieve correlation between the components greater than 0.5.

Key words: Farlie-Gumbel-Morgenstern class of distributions, characterization, symmetry and dependence, correlation structure, admissible range

1. Introduction

The so-called (bivariate) Farlie-Gumbel-Morgenstern (FGM) class of distributions originally introduced by Morgenstern (1956) for Cauchy marginal is an *important and efficient in applications*, class of multivariate distributions with given marginals. This structure investigated by Gumbel (1960) for exponential marginals and further generalized by Farlie (1960). Johnson and Kotz (1975), (1977) studied the multivariate case and presented detailed analysis of probabilistic and statistical characteristics. Huang and Kotz (1984) extended the bivariate FGM distribution in their attempts to increase the dependence between the underlying variables by introducing an additional parameter. The present paper provides some new properties of the generalized FGM family and introduces further generalizations which allow us to increase the dependence between the components. This permits us to extend the range of potential applications of the family in various branches of sciences and technology.

2. On the Huang-Kotz FGM distributions

Let (X, Y) be a bivariate absolutely continuous random variable with the distribution function

$$C_\alpha(x, y) = F(x)G(y)\{1 + \alpha A(F(x))B(G(y))\}, \tag{2.1}$$

where $A(x) \to 0$ and $B(x) \to 0$ as $x \to 1$ and the "kernels" $A(x), B(x)$ are differentiable functions on unit interval so that C_α becomes a distribution function with absolutely continuous marginals $F(x)$ and $G(y)$. The admissible range of α depends on the functions A and B. If $A(x) = B(x) = (1 - x)$, $0 \le x \le 1$, we then have the classical one parameter FGM family of distributions. If the marginals are uniform then for $A(x) = (1 - x)^\gamma$, $B(x) = (1 - x)^\gamma$, $\gamma \ge 1$ or $A(x) = 1 - x^\gamma$, $B(x) = 1 - x^\gamma$, $\gamma \ge \frac{1}{2}$, we have so-called symmetric modified Huang-Kotz *FGM* distributions. For $A(x) = 1 - x^{\gamma_1}$, $B(x) = 1 - x^{\gamma_2}$, $\gamma_1 \ne \gamma_2$ or $A(x) = (1 - x)^{\gamma_1}$, $B(x) = (1 - x)^{\gamma_2}$ we obtain an asymmetric modified Huang-Kotz *FGM* distribution. Admissible ranges of α for an extended distribution of the type $H_\alpha(x, y) = xy(1 + \alpha(1 - x^p)(1 - y^p))$, $p > 0$ and $H'_\alpha(x, y) = xy(1 + \alpha(1 - x)^p(1 - y)^p)$, $p > 0$ have been investigated in detail by Huang and Kotz (1999). For example, the admissible range of α for $H_\alpha(x, y)$ is

$$-(\max\{1, p\})^{-2} \le \alpha \le p^{-1}.$$

While for the classical FGM the correlation between components does not exceed $1/3$, the modified version allows correlation up to 0.39.

Let X_1, X_2 be independent and identically distributed continuous random variables. Then

$$P\{X_1 < X_2\} = \frac{1}{2}.$$

Evidently for (X, Y) possessing the bivariate distribution (2.1) with $A(x) = B(x)$ and $F(x) = G(x)$ the relationship

$$P\{X < Y\} = \frac{1}{2}$$

is valid. In fact, for uniform $[0, 1]$ marginals and for $A(x) = B(x)$ the joint density function is

$$c_\alpha(x, y) = 1 + \alpha(A(x) + xA'(x))(A(y) + yA'(y)),$$

where $A'(x) = \dfrac{dA(x)}{dx}$. One obtains

$$\begin{aligned} P\{X < Y\} &= P\{F(X) < F(Y)\} \\ &= \int_0^1 \int_0^y \{1 + \alpha(A(x) + xA'(x))(A(y) + yA'(y))\}\, dx\, dy \\ &= \frac{1}{2} + \alpha \int_0^1 \int_0^y \frac{d}{dx}(xA(x)) \frac{d}{dy}(yA(y))\, dx\, dy \\ &= \frac{1}{2} + \alpha \int_0^1 (yA(y))\, d(yA(y)) = \frac{1}{2}. \end{aligned}$$

Denote now

$$W(x) = \begin{vmatrix} A(x) & B(x) \\ A'(x) & B'(x) \end{vmatrix} = A(x)B'(x) - A'(x)B(x).$$

Definition 1: We say that FGM distribution (2.1) is in the class $\Im$ if the pair of kernels $A(x)$ and $B(x)$ are continuously differentiable and satisfy the conditions:

1° $A(0) = 1$ and $B(0) = 1$.

2° Either $W(x) \geq 0$ for all $0 \leq x \leq 1$ or $W(x) \leq 0$ for all $0 \leq x \leq 1$.

The following statements are valid.

Theorem 1. *Let (X, Y) be a continuous bivariate random variable with the distribution function*

$$C_\alpha(x, y) = F(x)F(y)\{1 + \alpha A(F(x))B(F(y))\},$$

where $A(x) \to 0$ and $B(x) \to 0$ as $x \to 1$ and the kernels $A(x), B(x)$ are differentiable functions on the unit interval so that C_α becomes a distribution function with absolutely continuous marginals. Suppose furthermore that $C_\alpha(x, y)$ is in the class $\Im$. Then $A(x) = B(x)$ if and only if

$$P\{X < Y\} = \frac{1}{2}.$$

Proof. We can assume without loss of generality that $F(x) = x$, $0 \leq x \leq 1$. The joint density is then

$$c_\alpha(x, y) = 1 + \alpha(A(x) + xA'(x))(B(y) + yB'(y)).$$

We have

$$\begin{aligned} P\{X < Y\} &= \int_0^1 \int_0^y \{1 + \alpha(A(x) + xA'(x))(B(y) + yB'(y))\}\, dx\, dy \\ &= \frac{1}{2} + \alpha \int_0^1 \int_0^y \frac{d}{dx}(xA(x))(B(y) + yB'(y))\, dx\, dy \\ &= \frac{1}{2} + \alpha \int_0^1 \Bigg(yA(y)B(y) + \frac{y^2}{2}(A(y)B(y))' \\ &\qquad + \frac{y^2}{2}(A(y)B'(y) - A'(y)B(y)) \Bigg) dy \\ &= \frac{1}{2} + \alpha \int_0^1 \Bigg(\frac{d}{dy}\left\{\frac{y^2}{2}A(y)B(y)\right\} \\ &\qquad + \frac{y^2}{2}(A(y)B'(y) - A'(y)B(y)) \Bigg) dy. \end{aligned} \tag{2.2}$$

It is easy to verify that $\int_0^1 \frac{d}{dy}\left\{\frac{y^2}{2}A(y)B(y)\right\}dy = 0$ for $C_\alpha(x, y) \in \Im$ and from (2.2) it follows that

$$P\{X < Y\} = \frac{1}{2} + \alpha \int_0^1 \frac{y^2}{2}(A(y)B'(y) - A'(y)B(y))\,dy. \tag{2.3}$$

Let

$$P\{X < Y\} = \frac{1}{2}.$$

Since the underlying distribution belongs to $\Im$, using (2.3) we have $A(y) = B(y)$. A proof of the "only if" part is straightforward.

Evidently the condition $A(x) = B(x)$ introduces symmetry between the variables X and Y.

Corollary 1. *Let (X, Y) be a continuous bivariate random variable with distribution function*

$$C_\alpha(x, y) = xy\{1 + \alpha(1 - x)^{\gamma_1}(1 - y)^{\gamma_2}\}.$$

The assertion $\gamma_1 = \gamma_2$ is valid if and only if

$$P\{X < Y\} = \frac{1}{2}.$$

Proof. Consider $A(x) = (1 - x)^{\gamma_1}$ and $B(x) = (1 - x)^{\gamma_2}$. We then have

$$A(y)B'(y) - A'(y)B(y) = (\gamma_1 - \gamma_2)(1 - y)^{\gamma_2+\gamma_1-1}\begin{cases} \geq 0 & \text{if } \gamma_1 \geq \gamma_2 \\ \leq 0 & \text{if } \gamma_1 \leq \gamma_2 \end{cases}$$

for all $0 \leq y \leq 1$. An application of Theorem 1 completes the proof.

Denote the conditional expectation $\varphi(x) = E(Y \mid X = x)$. Then for an (X, Y) having distribution function $C_\alpha(x, y) = xy\{1 + \alpha A(x)B(y)\}$,

$$\begin{aligned}
\varphi(x) = E(Y \mid X = x) &= \int_0^1 y\{1 + \alpha(A(x) + xA'(x))(B(y) + yB'(y))\}\,dy \\
&= \frac{1}{2} + \alpha(A(x) + xA'(x))\int_0^1 y(B(y) + yB'(y))\,dy \\
&= \frac{1}{2} + \alpha\frac{d}{dx}(xA(x))\int_0^1 y\,d\{yB(y)\} \\
&= \frac{1}{2} + \alpha\frac{d}{dx}(xA(x))\left\{-\int_0^1 yB(y)\,dy\right\}.
\end{aligned} \tag{2.4}$$

Analogously, denoting $\psi(x) = E(X \mid Y = x)$, one can write

$$E(X \mid Y = x) = \frac{1}{2} + \alpha \frac{d}{dx}(xB(x))\left\{-\int_0^1 yA(y)\,dy\right\}. \tag{2.5}$$

Utilizing (2.4) and (2.5) one easily proves the following

Theorem 1A. *Let (X, Y) be continuous bivariate random variable with distribution function*

$$C_\alpha(x, y) = xy\{1 + \alpha A(x)B(y)\},$$

where $A(x) \to 0$ and $B(x) \to 0$ as $x \to 1$ and $A(x), B(x)$ satisfy certain regularity conditions ensuring that C is a distribution function with uniform marginals. Assume that there exists a point $y_0 \in (0, 1)$ such that $A(y_0) = B(y_0)$. Then $A(x) = B(x)$ for all $0 \leq x \leq 1$ if and only if $\varphi(x) = \psi(x)$ for all $0 < x < 1$.

Proof. Let $\varphi(x) = \psi(x)$ for all $0 < x < 1$. Then from (2.4) and (2.5) one obtains

$$I_2 \frac{d}{dx}(xB(x)) = I_1 \frac{d}{dx}(xA(x)), \tag{2.6}$$

where $I_1 = \int_0^1 yA(y)\,dy$ and $I_2 = \int_0^1 yB(y)\,dy$. Integrating (2.6) over $(0, t)$, $0 < t < 1$, we have $I_2 tB(t) = I_1 tA(t)$. By the assumption, $B(y_0) = A(y_0)$, hence $I_2 = I_1$ and thus $A(x) = B(x)$. The "only if" assertion is straightforward.

Consider now the expected value of the model $X \geq Y$, $E(X \mid X \geq Y)$ (the left truncated model). It directly follows that for independent and identically distributed random variables X and Y having *uniform* $[0, 1]$ distribution,

$$E(X \mid X \geq Y) = \frac{2}{3}.$$

Indeed, for continuous random variables having joint density function $f(x, y)$, the conditional distribution of X under the condition $X \geq Y$ is

$$\begin{aligned} P\{X \leq x \mid X \geq Y\} &= \frac{P\{X \leq x, X \geq Y\}}{P\{X \geq Y\}} = \frac{P\{Y \leq X \leq x\}}{P\{X \geq Y\}} \\ &= \frac{1}{P\{X \geq Y\}} \int_a^x \int_a^u f(u, v)\,du\,dv, \end{aligned} \tag{2.7}$$

where $[a, b]$ is the support of X and Y, $-\infty \leq a < b \leq \infty$. Obviously the conditional density function of X under the condition $X \geq Y$ is

$$f_{X \mid X \geq Y}(x) = \frac{1}{P\{X \geq Y\}} \int_a^x f(x, v)\,dv. \tag{2.8}$$

The conditional expectation of X can thus be written as

$$E(X \mid X \geq Y) = \frac{1}{P\{X \geq Y\}} \int_a^b x \left\{ \int_a^x f(x, v)\, dv \right\} dx. \tag{2.9}$$

If X and Y are independent and identically distributed with the density $f(.)$, then using (2.8) we obtain

$$f_{X \mid X \geq Y}(x) = 2 \int_a^x f(x) f(v)\, dv = 2f(x)F(x) \tag{2.10}$$

and utilizing (2.9) we have

$$E(X \mid X \geq Y) = 2 \int_a^b xF(x)\, dF(x). \tag{2.11}$$

Now for the uniform $[0, 1]$ case, $F(x) = x$, $0 \leq x \leq 1$, from (2.11) we obtain that $E(X \mid X \geq Y) = \frac{2}{3}$.

Theorem 2. *Let (X, Y) be a bivariate continuous random variable possessing distribution function $C_\alpha(x, y) = xy\{1 + \alpha A(x)B(y)\}$ with uniform marginals. Let $C_\alpha(x, y) \in \Im$. Then $A(x) = B(x)$ for all $0 \leq x \leq 1$ if and only if $c_1 E(X \mid X \geq Y) = c_2 E(Y \mid Y \geq X)$, where $c_1 \equiv P\{X \geq Y\}$ and $c_2 \equiv P\{Y \geq X\}$.*

Proof. For (X, Y) having the joint density function $c_\alpha(x, y) = 1 + \alpha(A(x) + xA'(x))(B(y) + yB'(y))$ one obtains by means of straightforward calculations from (2.9) that

$$\begin{aligned} E(X \mid X \geq Y) &= \frac{1}{P\{X \geq Y\}} \int_0^1 x \left\{ \int_0^x \left[1 + \alpha \frac{d}{dx}(xA(x)) \frac{d}{dv}(vB(v)) \right] dv \right\} dx \\ &= \frac{1}{3c_1} + \frac{\alpha}{c_1} \int_0^1 x^2 B(x)\, d(xA(x)) \\ &= \frac{1}{3c_1} + \frac{\alpha}{c_1} \int_0^1 [x^2 B(x)A(x) + x^3 B(x)A'(x)]\, dx. \end{aligned} \tag{2.12}$$

Analogously, we have

$$E(Y \mid Y \geq X) = \frac{1}{3c_2} + \frac{\alpha}{c_2} \int_0^1 [x^2 B(x)A(x) + x^3 B'(x)A(x)]\, dx. \tag{2.13}$$

Necessity. Let $A(x) = B(x)$, then $c_1 = c_2 = \frac{1}{2}$ and by the symmetry

$$E(X \mid X \geq Y) = E(Y \mid Y \geq X),$$

i.e. $c_1 E(X \mid X \geq Y) = c_2 E(Y \mid Y \geq X)$.

Sufficiency. Let $c_1E(X \mid X \geq Y) = c_2E(Y \mid Y \geq X)$. Then one obtains from (2.12) and (2.13) that

$$\frac{1}{3} + \int_0^1 (y^2A(y)B(y)\,dy + y^3A(y)B'(y))\,dy$$
$$= \frac{1}{3} + \int_0^1 (y^2A(y)B(y) + y^3A'(y)B(y))\,dy$$

and

$$\int_0^1 y^3(A(y)B'(y) - A'(y)B(y))\,dy = 0.$$

Therefore $A(x) = B(x)$ for all $0 \leq x \leq 1$ as claimed.

Let $A(x) = B(x)$ for all $0 \leq x \leq 1$. One can write from (2.12)

$$\begin{aligned} E(X \mid X \geq Y) &= \frac{2}{3} + \alpha \int_0^1 [2x^2A^2(x) + 2x^3A(x)A'(x)]\,dx \\ &= \frac{2}{3} + \alpha \int_0^1 [3x^2A^2(x) + 2x^3A(x)A'(x)]\,dx - \alpha \int_0^1 x^2A^2(x)\,dx \\ &= \frac{2}{3} + \alpha \int_0^1 (x^3A^2(x))'\,dx - \alpha \int_0^1 x^2A^2(x)\,dx \\ &= \frac{2}{3} - \alpha \int_0^1 x^2A^2(x)\,dx. \end{aligned} \tag{2.14}$$

By symmetry

$$E(Y \mid Y \geq X) = \frac{2}{3} - \alpha \int_0^1 y^2A^2(y)\,dy. \tag{2.15}$$

2.1. A measure of dependence

In this section we propose an alternative simple measure of dependence for FGM distributions and their generalizations. The measure is motivated by

Theorem 3. *Let (X, Y) be a bivariate random variable with the joint distribution function*

$$C_\alpha(x, y) = xy\{1 + \alpha A(x)A(y)\}, \quad 0 \leq x, y \leq 1.$$

Then X and Y are independent if and only if

$$E(X \mid X \geq Y) = \frac{2}{3}.$$

Proof. For the independent X and Y with uniform marginals it follows from (2.11) that $E(X \mid X \geq Y) = \frac{2}{3}$. The only if assertion of the Theorem immediately follows from the equality (2.14)

$$E(X \mid X \geq Y) = \frac{2}{3} - \alpha \int_0^1 x^2 A^2(x)\, dx \equiv \frac{2}{3} - \alpha\rho^*, \tag{2.16}$$

where $\rho^* = \int_0^1 x^2 A^2(x)\, dx$. If $E(X \mid X \geq Y) = \frac{2}{3}$ then from the equation (2.16) we have $\alpha\rho^* = 0$. If $\alpha \neq 0$ is fixed then $\rho^* = 0$, which implies $A \equiv 0$.

Consequently for random variables (X, Y) with joint distribution function given as in Theorem 3 the value

$$D_\alpha(X, Y) = \left| E(X \mid X \geq Y) - \frac{2}{3} \right| = |\alpha|\rho^*,$$

where $\rho^* = \int_0^1 x^2 A^2(x)\, dx$, can be used to characterize the dependence between X and Y for a fixed admissible α. The case with $D_\alpha(X, Y) = 0$ corresponds to independent X and Y. If α is fixed and $D_\alpha(X, Y) \to 0$ then $\rho^* \to 0$ and hence $A(x) \to 0$. We shall provide several examples. Let α be fixed.

1. For $A(x) = 1 - x^p$, (the Huang-Kotz FGM 1) we have

$$\rho_1^*(p) \equiv \rho_1^* = \int_0^1 x^2 (1 - x^p)^2\, dx = \frac{1}{3} - \frac{2}{p+3} + \frac{1}{2p+3}. \tag{2.17}$$

If $p = 1$ we have from (2.17) $\rho_1^* = \frac{1}{30}$. If $p \downarrow 0$ then $\rho_1^* \downarrow 0$ (see Figure 1(a)). (For this distribution small values of p would seem to correspond to a weak dependence of X and Y.)

2. Let $A(x) = (1 - x)^p$, $0 < x < 1$, $p > 0$. Then

$$\rho_2^*(p) \equiv \rho_2 = \int_0^1 x^2 (1 - x)^{2p}\, dx = Beta(3, 2p + 1) \tag{2.18}$$

and for an integer p one could write $\rho_2^*(p) = \dfrac{2!(2p)!}{(3 + 2p)!}$.

Taking $p = 2$ in (2.18) we have $\rho_2^* = 1/105$, taking $p = 3$ we arrive at $\rho_2^* = 1/252$. It seems that if $p \uparrow$ then $\rho_2^* \downarrow 0$ (see Figure 1(b)). (For this distribution large value of p would before seem to correspond to a weak dependence between X and Y.)

3. Let $A(x) = (1 - x^2)^p$, $p > 1$. We then have a distribution of the form

$$F_p(x, y) = xy\{1 + \alpha(1 - x^2)^p (1 - y^2)^p\}, \quad p > 1,\ 0 < x, y < 1 \tag{2.19}$$

with the joint density

$$f_p(x, y) = 1 + \alpha(1 - x^2)^{p-1}(1 - y^2)^{p-1}[1 - x^2(1 + 2p)][1 - y^2(1 + 2p)].$$

It can be verified that the admissible range of α for this distribution is

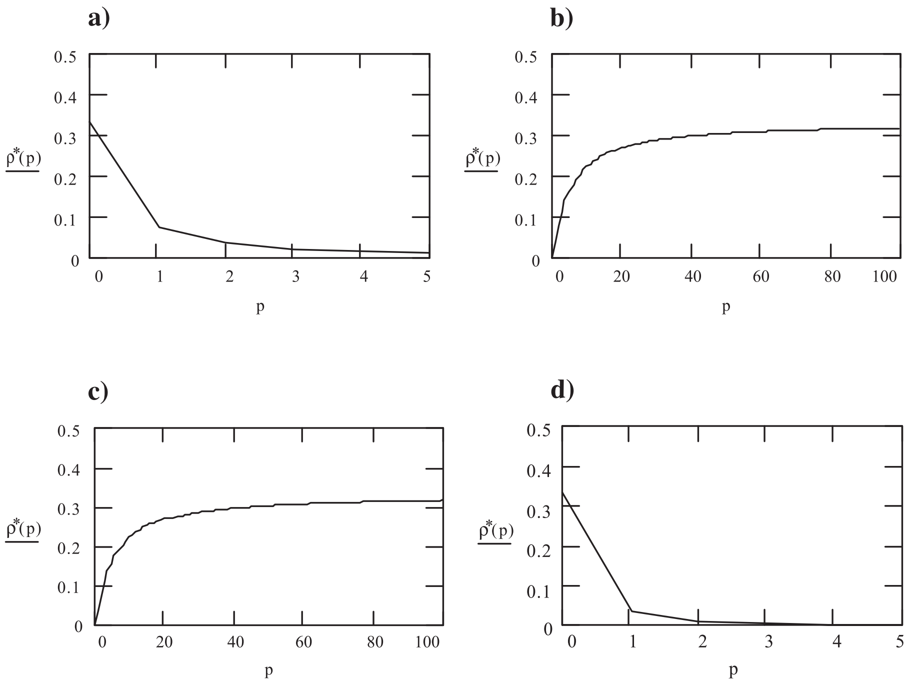

Fig. 1. Graphs of $\rho_i^*(p),\ i = 1, 2, 3, 4.$

$$-\min\left\{\frac{1}{4}\frac{1}{\left(1-\dfrac{3}{1+2p}\right)^{2(p-1)}}, 1\right\} \leq \alpha \leq \frac{1}{2}\frac{1}{\left(1-\dfrac{3}{1+2p}\right)^{p-1}}$$

(The discussion of the admissible range for α is provided in Section 3). If $p = 1$ one has the distribution

$$F(x, y) = xy\{1 + \alpha(1 - x^2)(1 - y^2)\}, \quad 0 < x, y < 1$$

with the joint density

$$f_p(x, y) = 1 + \alpha[1 - 3x^2][1 - 3y^2].$$

For distribution (2.19)

$$\rho_3^* \equiv \rho_3^*(p) = \int_0^1 x^2(1 - x^2)^{2p}\, dx = \frac{1}{2} Beta\left(\frac{3}{2}, 2p + 1\right).$$

If $p \to \infty$ then $\rho_3^*(p) \to 0$. (see Figure 1(c)).

4. Let $A(x) = (1 - x^p)^2$, $p > 0$. Then one has the distribution of the form

Table 1. Some selected values of $\rho_i^*(p)$, $i = 1, 2, 3, 4$

p	$\rho_1^*(p)$	$\rho_2^*(p)$	$\rho_3^*(p)$	$\rho_4^*(p)$
1/4	0.004	0.152	0.196	0.0001
1/3	0.006	0.123	0.171	0.0004
1/2	0.012	0.083	0.133	0.002
2/3	0.009	0.059	0.108	0.004
1	0.033	0.033	0.076	0.01
3/2	0.056	0.017	0.051	0.022
4/3	0.048	0.021	0.057	0.018
2	0.076	0.01	0.037	0.037
3	0.111	0.004	0.023	0.067
4	0.139	0.002	0.016	0.093
5	0.16	0.001	0.012	0.116
10	0.223	0.0001	0.005	0.189
20	0.27	0.00002	0.002	0.248
40	0.299	0.000003	0.0006	0.286
100	0.319	0.00000007	0.0001	0.313

$$F_p(x, y) = xy\{1 + \alpha(1 - x^p)^2(1 - y^p)^2\}, \quad p > 0,\ 0 < x, y < 1 \tag{2.20}$$

with the joint density

$$f_p(x, y) = 1 + \alpha(1 - x^p)(1 - y^p)[1 - x^p(1 + 2p)] \\ \times [1 - y^p(1 + 2p)], \quad p > 0,\ 0 \le x, y \le 1.$$

The admissible range of α can be investigated analogously (see Section 3).

For distribution with the distribution function (2.20) one has

$$\rho_4^* \equiv \rho_4^*(p) = \int_0^1 x^2(1 - x^p)^4\, dx = \frac{1}{p} Beta\left(\frac{3}{p}, 3\right).$$

If $p \to 0$, $\rho_4^* \to 0$. (see Figure 1(d)).

As it follows from the examples above that for the symmetric Huang-Kotz FGM class of distributions and their extensions with uniform marginals, the cases of $\rho^* \downarrow 0$ correspond to a weak dependence. Some selected values of ρ_1^*, ρ_2^*, ρ_3^* and ρ_4^* are presented in Table 1.

Remark. The proposed measure is expressed as a deviation between $E(X \mid X \geq Y) = E(X)$ in the independence case (in our case also equal to $E(Y)$) and the same function $E(X \mid X \geq Y)$ in the case of dependence. It does provide a natural measure of dependence (different from correlation coefficient) quite suitable for FGM type distributions. This measure may merit further investigation.

3. Modifications of the Huang-Kotz FGM distributions

1. We shall now consider several modifications of the Huang-Kotz distribution. The structure of distributions discussed seems to be quite natural in the sense that they are recursively built upon the independence structure by adding terms of the form $\alpha A(x)B(y)$. We shall concentrate in this section on the form of the type $(1 - x^q)^p$ which are easily implemented and estimated allowing to reach correlation exceeding 0.5. This fact partially eliminates the objections against FGM distributions which originally would not allow correlation above one third.

Let

$$F_{p,\alpha}(x, y) = xy\{1 + \alpha(1 - x^2)^p(1 - y^2)^p\}, \quad p > 0,\ 0 < x, y < 1. \tag{3.1}$$

Case $p = 1$ corresponds to the Huang-Kotz distribution. One easily verifies that the joint density function of (3.1) is

$$f_{p,\alpha}(x, y) = 1 + \alpha(1 - x^2)^{p-1}(1 - y^2)^{p-1}[1 - x^2(1 + 2p)][1 - y^2(1 + 2p)]. \tag{3.2}$$

We shall now investigate the admissible range of α for distribution with distribution function (3.1). The argument is similar to one presented in Huang-Kotz (1999).

The overall constraint on α is given by

$$\alpha(1 - x^2)^{p-1}(1 - y^2)^{p-1}[1 - x^2(1 + 2p)][1 - y^2(1 + 2p)] \geq -1$$

Let $0 < p < 1$. On one hand we have

$$\alpha \geq -\frac{(1 - x^2)^{1-p}(1 - y^2)^{1-p}}{[1 - x^2(1 + 2p)][1 - y^2(1 + 2p)]}.$$

The value 0 is attained at $x = y = 1$ and -1 is attained at $x = y = 0$, yielding the lower bound $\alpha \geq 0$. On the other hand

$$\alpha \leq \frac{(1 - x^2)^{1-p}(1 - y^2)^{1-p}}{[1 - x^2(1 + 2p)][y^2(1 + 2p) - 1]}.$$

The value 0 is attained at the point $x = 0,\ y = 1$. Thus $0 \leq \alpha \leq 0$, and any $\alpha \neq 0$ is inadmissible for $p < 1$. We shall thus deal with the case $p > 1$ only.

It is clear that $f_p(x, y) = 1$ on the lines $y = \left(\frac{1}{1+2p}\right)^{1/2}$, $x = \left(\frac{1}{1+2p}\right)^{1/2}$.
Consider the following four cases.

1. In $Q1 = \left\{(x, y)\colon \frac{1}{\sqrt{1+2p}} < x, y < 1\right\}$ we are required to have

$1 + \alpha(1 - x^2)^{p-1}(1 - y^2)^{p-1}[x^2(1 + 2p) - 1][y^2(1 + 2p) - 1] \geq 0,$ i.e.

$$\alpha \geq -\frac{1}{(1 - x^2)^{p-1}(1 - y^2)^{p-1}[x^2(1 + 2p) - 1][y^2(1 + 2p) - 1]}. \quad (3.3)$$

Consider the function

$$r(x) = (1 - x^2)^{p-1}[x^2(1 + 2p) - 1], \quad \frac{1}{\sqrt{1+2p}} < x < 1.$$

One easily checks that

$$r'(x) = (1 - x^2)^{p-2}2xp[3 - x^2(1 + 2p)]. \quad (3.4)$$

The solution of $r'(x) = 0$, i.e. the point $x_* = \left(\frac{3}{1+2p}\right)^{1/2}$ is an extreme point of $r(x)$. Differentiating (3.4) we arrive at

$$r''(x) = (1 - x^2)^{p-3}p[2(1 - x^2) - (p - 2)4x^2][3 - (1 + 2p)x^2]$$
$$- (1 - x^2)^{p-2}4x^2(1 + 2p)p \quad (3.5)$$

From (3.5) it follows that

$$r''\left(\left(\frac{3}{1+2p}\right)^{1/2}\right) = -12\left(1 - \frac{3}{1+2p}\right)^{p-2}p < 0, \quad (p > 1).$$

Therefore at the point $x_* = \left(\frac{3}{1+2p}\right)^{1/2}$ $r(x)$ attains its maximum. One can write

$$r\left(\left(\frac{3}{1+2p}\right)^{1/2}\right) = 2\left(1 - \frac{3}{1+2p}\right)^{2(p-1)}.$$

Thus in $Q1$ one has

$$\alpha \geq -\frac{1}{4}\frac{1}{\left[\left(1 - \frac{3}{1+2p}\right)^{2(p-1)}\right]} \quad (3.6)$$

Using l'Hôpital's rule we have

$$\lim_{p\to 1}\left[\left(1-\frac{3}{1+2p}\right)^{2(p-1)}\right] = \lim_{p\to 1}\exp\left\{2(p-1)\ln\left(1-\frac{3}{1+2p}\right)\right\}$$

$$= \lim_{p\to 1}\exp\left\{\frac{2\ln\left(1-\frac{3}{1+2p}\right)}{\frac{1}{p-1}}\right\}$$

$$= \lim_{p\to 1}\exp\left\{\frac{6(p-1)}{(1+2p)}\right\} = 1.$$

Thus $\alpha \geq -\frac{1}{4}$, for $p = 1$.

2. In $Q2 = \left\{(x, y)\colon 0 < y < \frac{1}{\sqrt{1+2p}} < x < 1\right\}$. We have

$$1-\alpha(1-x^2)^{p-1}(1-y^2)^{p-1}[x^2(1+2p)-1][1-y^2(1+2p)] \geq 0,$$

or equivalently

$$\alpha \leq \frac{1}{(1-x^2)^{p-1}(1-y^2)^{p-1}[x^2(1+2p)-1][1-y^2(1+2p)]}.$$

One can verify that

$$\min_{(x,y)\in Q2}\left(\frac{1}{(1-x^2)^{p-1}(1-y^2)^{p-1}[x^2(1+2p)-1][1-y^2(1+2p)]}\right)$$

$$=\frac{1}{2}\left(1-\frac{3}{1+2p}\right)^{-(p-1)},$$

attained at the point $x = \left(\frac{3}{1+2p}\right)^{1/2}$, $y = 0$. Thus in $Q2$ one has

$$\alpha \leq \frac{1}{2}\left(1-\frac{3}{1+2p}\right)^{-(p-1)}.$$

3. In $Q4 = \left\{(x, y) : (x, y)\colon 0 < x < \frac{1}{\sqrt{1+2p}} < y < 1\right\}$ by the analogy with $Q2$,

$$\alpha < \frac{1}{2}\left(1-\frac{3}{1+2p}\right)^{-(p-1)}.$$

4. In $Q3 = \left\{(x, y): 0 < x, y < \frac{1}{\sqrt{1+2p}}\right\}$ we are required to have

$$\alpha \geq -\frac{1}{(1-x^2)^{p-1}(1-y^2)^{p-1}[1-x^2(1+2p)][1-y^2(1+2p)]}$$

and the minimum attained at $x = y = 0$. Hence in $Q3,\ \alpha \geq -1$.
The admissible range for α is thus:

$$-\min\left\{\frac{1}{4}\left(\frac{1+2p}{2(p-1)}\right)^{2(p-1)}, 1\right\} \leq \alpha \leq \frac{1}{2}\left(\frac{1+2p}{2(p-1)}\right)^{p-1}.$$

Some selected values for $\alpha_{\min}$ and $\alpha_{\max}$ are presented below:

$p = 1$	$\alpha_{\min} = -0.25$	$p = 1.00$	$\alpha_{\max} = 0.50$	$p = 2$	$\alpha_{\max} = 1.25$
$p = 1.05$	$\alpha_{\min} = -0.36$	$p = 1.01$	$\alpha_{\max} = 0.53$	$p = 10$	$\alpha_{\max} = 2.01$
$p = 1.20$	$\alpha_{\min} = -0.60$	$p = 1.05$	$\alpha_{\max} = 0.59$	$p = 100$	$\alpha_{\max} = 2.20$
$p = 1.49$	$\alpha_{\min} = -0.99$	$p = 1.10$	$\alpha_{\max} = 0.69$	$p = 1000$	$\alpha_{\max} = 2.23$
$p = 1.50$	$\alpha_{\min} = -1.00$	$p = 1.50$	$\alpha_{\max} = 1.00$	$p = 10000$	$\alpha_{\max} = 2.28$

1.A. Consider now the following generalization of (3.1):

$$F_{n,p,\alpha}(x, y) = xy\{1 + \alpha(1-x^n)^p(1-y^n)^p\}, \quad p > 0,\ 0 < x, y < 1,\ n \geq 1.$$

Arguments analogous to those in Section 1 show that the joint probability density this distribution is

$$f_{n,p,\alpha}(x, y) = 1 + \alpha(1-x^n)^{p-1}(1-y^n)^{p-1}[1-x^n(1+np)][1-y^n(1+np)],$$
$$p > 0,\ 0 \leq x, y \leq 1,\ n \geq 1.$$

The overall constraint on α is given by

$$\alpha(1-x^n)^{p-1}(1-y^n)^{p-1}[1-x^n(1+np)][1-y^n(1+np)] \geq -1$$

Let $0 < p < 1$. On one hand we have

$$\alpha \geq -\frac{(1-x^n)^{1-p}(1-y^n)^{1-p}}{[1-x^n(1+np)][1-y^n(1+np)]},$$

the value $\alpha = 0$ is attained at $x = y = 1$ and $= -1$ at $x = y = 0$, yielding the lower bound $\alpha \geq 0$. On the other hand,

$$\alpha \leq \frac{(1-x^n)^{1-p}(1-y^n)^{1-p}}{[1-x^n(1+np)][y^n(1+np)-1]},$$

the value $\alpha = 0$ attained at $x = 0$, $y = 1$. Thus $0 \leq \alpha \leq 0$, and hence any $\alpha \neq 0$ is inadmissible for $p < 1$. We shall therefore deal with the case $p > 1$ only.

It is evident that $f_p(x, y) = 1$ on the lines $y = \left(\frac{1}{1+np}\right)^{1/n}$, $x = \left(\frac{1}{1+np}\right)^{1/n}$. By the analogy with the case $n = 2$, denote

$$r(x) = (1-x^n)^{p-1}[x^n(1+np)-1], \quad \left(\frac{1}{1+np}\right)^{1/n} < x < 1.$$

One can easily check that the solution of the equation $r'(x) = 0$ i.e. $x_* = \left(\frac{n+1}{1+np}\right)^{1/n}$ is the point of maximum of $r(x)$, since $r''(x_*) < 0$. Repeating the arguments presented in Section 1 one can verify that the admissible range for α is:

$$-\min\left\{\frac{1}{n^2}\left(\frac{1+np}{n(p-1)}\right)^{2(p-1)}, 1\right\} \leq \alpha \leq \frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}.$$

For $p = 1$ one has the Huang-Kotz (1999) case 1 and

$$-(\max\{1, n\})^{-2} \leq \alpha \leq \frac{1}{n}$$

For $n = 1$ one has the Huang-Kotz (1999) case 2 and

$$-1 \leq \alpha \leq \left(\frac{p+1}{p-1}\right)^{p-1}$$

(see Huang-Kotz (1998)). Some selected values for $\alpha_{\min}$ and $\alpha_{\max}$ are presented below:

	for $n = 3$		*for* $n = 3$	*for* $n = 4$
$p = 1.05$	$\alpha_{\min} = -0.15$	$p = 1.01,$	$\alpha_{\max} = 0.35$	$\alpha_{\max} = 0.26$
$p = 1.20$	$\alpha_{\min} = -0.25$	$p = 1.05,$	$\alpha_{\max} = 0.39$	$\alpha_{\max} = 0.29$
$p = 1.50$	$\alpha_{\min} = -0.41$	$p = 1.10,$	$\alpha_{\max} = 0.43$	$\alpha_{\max} = 0.32$
$p = 2.50$	$\alpha_{\min} = -0.75$	$p = 1.50$	$\alpha_{\max} = 0.64$	$\alpha_{\max} = 0.46$
$p = 3.50$	$\alpha_{\min} = -0.94$	$p = 2.00$	$\alpha_{\max} = 0.78$	$\alpha_{\max} = 0.56$
$p = 3.80$	$\alpha_{\min} = -0.99$	$p = 10.0$	$\alpha_{\max} = 1.16$	$\alpha_{\max} = 0.80$
$p = 3.99$	$\alpha_{\min} = -1.00$	$p = 100$	$\alpha_{\max} = 1.25$	$\alpha_{\max} = 0.87$

Using the Hoeffding formula (see, e.g., Lehmann (1966))

$$cov(X, Y) = \iint [F(x, y) - F_X(x)F_Y(y)]\, dx\, dy$$

one observes that for $F_{n,p,\alpha}(x, y)$ the correlation coefficient is equal to

$$\rho \equiv \rho(X, Y) = 12\alpha t^2(p, n)$$

where $t(x, y) \equiv \dfrac{\Gamma(x+1)\Gamma(2/y)}{y\Gamma(x+1+2/y)}$. The so called Schweizer-Wolff index of dependence defined as

$$\sigma(X, Y) = 12\int_0^1\int_0^1 |F(x, y) - xy|\, dx\, dy$$

for uniform marginals (see e.g. Schweizer (1991)) is $\sigma(X, Y) = 12|\alpha|t^2(p, n)$, coincides for these distributions (up to the sign) with the correlation coefficient. The range for correlation coefficient is

$$-12t^2(p, n)\min\left\{\frac{1}{n^2}\left(\frac{1+np}{n(p-1)}\right)^{2(p-1)}, 1\right\} \le \rho \le 12t^2(p, n)\frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}$$

For $n = 2$ one obtains the distribution given in (2.1). In this case $t(2, p) = 1/2(p+1)$ and $\rho = 12\alpha(2(p+1))^{-2}$. The range between the maximal and minimal values of ρ for a specific value of p is given by

$$-\frac{3}{(p+1)^2}\min\left\{\frac{1}{4}\frac{1}{\left(1-\dfrac{3}{1+2p}\right)^{2(p-1)}}, 1\right\}$$

$$\le \rho \le \frac{3}{2}\frac{1}{(p+1)^2}\frac{1}{\left(1-\dfrac{3}{1+2p}\right)^{(p-1)}}$$

A refined iterative search yields the maximum correlation $\rho_{\max}(p) = 0.501594$ at $n = 2.896772$, $p = 1.490834$; and the minimum $\rho_{\min}(p) = -0.48$ at $n = 2$, $p = 1.5$. The fact that $\rho_{\max}(p)$ exceeds 1/2 is gratifying for applications.

2. Consider now the following generalization of (3.1).

$$F_{p,q,n;\alpha}(x, y) = x^p y^p\{1 + \alpha(1 - x^q)^n(1 - y^q)^n\},$$

$$p, q \ge 1,\ n > 1,\ 0 < x, y < 1, \tag{3.11}$$

with marginals $F(x) = x^p$, $G(y) = y^p$, $0 < x, y < 1$. One easily checks that the probability density function of (3.11) is

$$f_{p,q,n;\alpha}(x, y) = x^{p-1}y^{p-1}\{p^2 + \alpha(1 - x^q)^{n-1}[(p - x^q(p + qn)](1 - y^q)^{n-1} \times [(p - y^q(p + qn)]\}, \quad p, q \geq 1,\ n > 1,\ 0 < x, y < 1.$$

Repeating arguments presented in Section 3 one can obtain the following admissible range for α:

$$-\min\left\{1, \frac{p^2}{q^2}\left[\frac{p + qn}{q(n-1)}\right]^{2(n-1)}\right\} \leq \alpha \leq \frac{p}{q}\left[\frac{p + qn}{q(n-1)}\right]^{n-1}$$

The admissible range for correlation coefficient is

$$-a(n, p, q)\min\left\{1, \frac{p^2}{q^2}\left[\frac{p + qn}{q(n-1)}\right]^{2(n-1)}\right\} \leq \rho \leq a(n, p, q)\frac{p}{q}\left[\frac{p + qn}{q(n-1)}\right]^{n-1},$$

where

$$a(n, p, q) = \left(\frac{p}{(p+2)(p+1)^2}\right)^{-1} \frac{1}{q^2}\left(Beta\left(\frac{p+1}{q}, n + 1\right)\right)^2.$$

For $p = 1$, $q = 2$ the maximal negative correlation $\rho_{\text{min}} = -0.4794$ is attained at the point $n = 1.495$. For $p = 0.001$, $q = 1.5$ the maximal positive correlation $\rho_{\text{max}} = 0.6122$ is attained at the point $n = 1.379$.

Conclusion

In this paper we have analyzed and extended the flexible and intuitively appealing bivariate Farlie-Gumbel-Morgenstern model (which possesses convenient statistical properties) and have shown that the correlation coefficient can exceed 1/2 by a judicious choice of parameters which renders it appropriate for modeling in various applications.

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