



Spacings, exceedances and concomitants in progressive type II censoring scheme

Ismihan Bairamov*, Serkan Eryılmaz

*Department of Mathematics, Izmir University of Economics, Sakarya Caddesi No. 156, 35330, Balçova, Izmir,
Turkey*

Received 28 October 2003; accepted 8 September 2004

Available online 27 October 2004

Abstract

Distributional properties of spacings and exceedances based on progressive type II censored order statistics are investigated. Recurrence relations between moments of concomitants for the model of progressive type II censoring are given. We unify and extend some results on spacings, exceedances and concomitants for the model of progressive censoring.

© 2004 Elsevier B.V. All rights reserved.

MSC: 62G30; 60E99

Keywords: Progressive type II censored order statistics; Generalized order statistics; Spacings; Exceedances; Recurrence relations

1. Introduction

The model of progressive type II censoring is of importance in the field of reliability and life testing. Suppose N identical units are placed on a lifetime test. At the time of i th failure, R_i surviving units are randomly withdrawn from the experiment, $1 \leq i \leq n$. Thus, if n failures are observed then $R_1 + \dots + R_n$ units are progressively censored; hence $N = n + R_1 + \dots + R_n$.

* Corresponding author. Tel.: +232 279 2525; fax: +232 279 2626.

E-mail address: ismihan.bayramoglu@ieu.edu.tr (I. Bairamov).

$X_{1:n}^{\tilde{R}} \leq X_{2:n}^{\tilde{R}} \leq \dots \leq X_{n:n}^{\tilde{R}}$ describe the progressively censored failure times. $\tilde{R} = (R_1, \dots, R_n)$ denotes the censoring scheme. As a special case if $\tilde{R} = (0, \dots, 0)$ where no withdrawals are made we obtain the ordinary order statistics.

Progressive censoring is discussed in the works of Viveros and Balakrishnan (1994), Kamps (1995), Balakrishnan and Sandhu (1996), Aggarwala and Balakrishnan (1996), and Kamps and Cramer (2001). Balakrishnan et al. (2001a) give bounds for means and variances of progressive type II censored order statistics. In the recent paper of Balakrishnan et al. (2001b) progressive type II censored order statistics from exponential distribution are considered.

If the failure times are from an absolutely continuous distribution function (d.f.) F and probability density function (p.d.f.) f , the joint p.d.f. of the progressive type II censored order statistics is given by

$$f^{X_{1:n}^{\tilde{R}}, \dots, X_{n:n}^{\tilde{R}}}(x_1, \dots, x_n) = c \prod_{j=1}^n f(x_j)(1 - F(x_j))^{R_j},$$

where $c = \prod_{j=1}^n \gamma_j$ with $\gamma_j = N - \sum_{i=1}^{j-1} R_i - j + 1$, $\gamma_1 = N$ and $N = n + \sum_{j=1}^n R_j$.

The marginal d.f. and p.d.f. of $X_{r:n}^{\tilde{R}}$ are given, respectively, by

$$F^{X_{r:n}^{\tilde{R}}}(x) = 1 - c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} (1 - F(x))^{\gamma_i}, \quad 1 \leq r \leq n$$

and

$$f^{X_{r:n}^{\tilde{R}}}(x) = c_{r-1} f(x) \sum_{i=1}^r a_{i,r} (1 - F(x))^{\gamma_i - 1}, \quad 1 \leq r \leq n, \tag{1}$$

where $c_{r-1} = \prod_{j=1}^r \gamma_j$ and $a_{i,r} = \prod_{\substack{j=1 \\ j \neq i}}^r (1/\gamma_j - \gamma_i)$, $1 \leq i \leq r \leq n$, $n \geq 2$, and the empty product \prod_{\emptyset} is defined to be 1 (Kamps and Cramer, 2001).

Progressive type II censored order statistics can be viewed as a special case of generalized order statistics which were introduced by Kamps (1995). In the present paper, we consider spacings from exponential distribution, exceedances and concomitants of progressive type II censored order statistics.

2. Spacings

Let X_1, X_2, \dots, X_n be a random sample with continuous d.f. F . Denote by $X_{1:n}, X_{2:n} - X_{1:n}, \dots, X_{n:n} - X_{n-1:n}$ the spacings (differences) of this sample. It is well known that if the underlying distribution is exponential with parameter λ , then $X_{k:n} - X_{k-1:n}$, $k = 1, 2, \dots, n$ ($X_{0:n} = 0$) are independent and $X_{k:n} - X_{k-1:n}$ has an exponential distribution with parameter $(n - k + 1)\lambda$.

Consider the progressive type II censored order statistics $X_{1:n}^{\tilde{R}} \leq X_{2:n}^{\tilde{R}} \leq \dots \leq X_{n:n}^{\tilde{R}}$ based on an absolutely continuous d.f. F . The independence property also holds for the random

variables $X_{1:n}^{\tilde{R}}, X_{2:n}^{\tilde{R}} - X_{1:n}^{\tilde{R}}, \dots, X_{n:n}^{\tilde{R}} - X_{n-1:n}^{\tilde{R}}$ in the case of exponential distribution (Kamps, 1995). In this case $X_{k:n}^{\tilde{R}} - X_{k-1:n}^{\tilde{R}}, k = 1, 2, \dots, n$ has an exponential distribution with parameter $\lambda\gamma_k$, where $\gamma_k = N - \sum_{i=1}^{k-1} R_i - k + 1, \gamma_1 = N$.

Define the following random variables:

$$v^{\tilde{R}} = k \text{ iff } X_{k:n}^{\tilde{R}} - X_{k-1:n}^{\tilde{R}} \leq X_{i:n}^{\tilde{R}} - X_{i-1:n}^{\tilde{R}}, \quad i = 1, 2, \dots, n; \quad i \neq k$$

and

$$\eta^{\tilde{R}} = l \text{ iff } X_{l:n}^{\tilde{R}} - X_{l-1:n}^{\tilde{R}} \geq X_{i:n}^{\tilde{R}} - X_{i-1:n}^{\tilde{R}}, \quad i = 1, 2, \dots, n; \quad i \neq l.$$

It is clear that $v^{\tilde{R}}$ and $\eta^{\tilde{R}}$ are the index of a spacing having minimal and maximal length, respectively. In this case the random variables $X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}}$ and $X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}}$ represent the minimal and maximal spacings of progressive type II censored order statistics, i.e. the extremes of spacings. Since the random variables $X_{1:n}^{\tilde{R}} \leq X_{2:n}^{\tilde{R}} \leq \dots \leq X_{n:n}^{\tilde{R}}$ can be viewed the failure times of units in progressive type II censoring scheme, extremes of spacings stand for minimum and maximum length between two failures. Therefore they play an important role to make a decision for a system. In this section, the joint and marginal distributions of the random variables $v^{\tilde{R}}, \eta^{\tilde{R}}, X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}}, X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}}$ are presented. We shall first establish the following lemma.

Lemma 2.1. Let Z_1, Z_2, \dots, Z_n be independent random variables and $P\{Z_i \leq x\} = F_i(x), i = 1, 2, \dots, n$. Define

$$\alpha \equiv \{k : Z_k < Z_i, \quad i \neq k\}$$

and

$$\beta \equiv \{l : Z_l > Z_i, \quad i \neq l\}.$$

It is true that

$$P\{\alpha = k, \beta = l\} = \int_{-\infty}^{\infty} \int_x^{\infty} \frac{\prod_{i=1}^n (F_i(y) - F_i(x))}{(F_l(y) - F_l(x))(F_k(y) - F_k(x))} dF_l(y) dF_k(x), \quad k \neq l.$$

Proof. By the definition of α and β

$$\begin{aligned} P\{\alpha = k, \beta = l\} &= P\{Z_k < Z_1, \dots, Z_k < Z_{k-1}, Z_k < Z_{k+1}, \dots, Z_k < Z_{l-1}, \\ &\quad Z_k < Z_{l+1}, \dots, Z_k < Z_n, Z_l > Z_1, \dots, Z_l > Z_{k-1}, \\ &\quad Z_l > Z_{k+1}, \dots, Z_l > Z_{l-1}, Z_l > Z_{l+1}, \dots, Z_l > Z_n\} \\ &= P\{Z_k < Z_1 < Z_l, \dots, Z_k < Z_{k-1} < Z_l, Z_k < Z_{k+1} < Z_l, \\ &\quad \dots, Z_k < Z_{l-1} < Z_l, Z_k < Z_{l+1} < Z_l, \dots, Z_k < Z_n < Z_l\} \end{aligned}$$

by conditioning on Z_k, Z_l the proof is completed. \square

Theorem 2.1. Let $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Then

$$P \left\{ v^{\tilde{R}} = k, \eta^{\tilde{R}} = l \right\} = \lambda^2 \gamma_l \gamma_k \int_0^\infty \int_x^\infty \frac{\prod_{i=1}^n (e^{-\lambda \gamma_i x} - e^{-\lambda \gamma_i y})}{(e^{-\lambda \gamma_l x} - e^{-\lambda \gamma_l y})(e^{-\lambda \gamma_k x} - e^{-\lambda \gamma_k y})} e^{-\lambda \gamma_l y} e^{-\lambda \gamma_k x} dy dx \quad (2)$$

and

$$P \left\{ v^{\tilde{R}} = k \right\} = \frac{\gamma_k}{\sum_{i=1}^n \gamma_i}, \quad P \left\{ \eta^{\tilde{R}} = l \right\} = \lambda \gamma_l \int_0^\infty \frac{\prod_{i=1}^n (1 - e^{-\lambda \gamma_i x})}{(1 - e^{-\lambda \gamma_l x})} e^{-\lambda \gamma_l x} dx, \quad (3)$$

$k \neq l$, $k, l = 1, 2, \dots, n$.

Proof. The equation given in (2) is obtained taking $F_i(x) = 1 - e^{-\lambda \gamma_i x}$, $i = 1, 2, \dots, n$ in Lemma 2.1. For marginal distributions one can write

$$P \left\{ v^{\tilde{R}} = k \right\} = P \left\{ X_{k:n}^{\tilde{R}} - X_{k-1:n}^{\tilde{R}} \leq X_{i:n}^{\tilde{R}} - X_{i-1:n}^{\tilde{R}}, i = 1, 2, \dots, n; i \neq k \right\} \quad (4)$$

and

$$P \left\{ \eta^{\tilde{R}} = l \right\} = P \left\{ X_{l:n}^{\tilde{R}} - X_{l-1:n}^{\tilde{R}} \geq X_{i:n}^{\tilde{R}} - X_{i-1:n}^{\tilde{R}}, i = 1, 2, \dots, n; i \neq l \right\} \quad (5)$$

conditioning on $X_{k:n}^{\tilde{R}} - X_{k-1:n}^{\tilde{R}}$ and $X_{l:n}^{\tilde{R}} - X_{l-1:n}^{\tilde{R}}$ in (4) and (5), respectively, and using the independence property of spacings the result is obtained. \square

In the following theorem, the joint and marginal distributions of extremes of spacings are given.

Theorem 2.2. Let $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Then

$$P \left\{ X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}} \leq x, X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}} \leq y \right\} = \prod_{i=1}^n (1 - e^{-\lambda \gamma_i y}) - \prod_{i=1}^n (e^{-\lambda \gamma_i x} - e^{-\lambda \gamma_i y}), \quad x < y.$$

Proof. It is obvious that

$$P \left\{ X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}} \leq x, X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}} \leq y \right\} = P \left\{ X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}} \leq y \right\} - P \left\{ X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}} > x, X_{\eta^{\tilde{R}}:n}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}} < y \right\}.$$

Repeating the arguments similar to the proof of Theorem 2.1 the result is obtained. \square

Corollary 2.1. Let $F(x) = 1 - e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Then

$$P \left\{ X_{v^{\tilde{R}}:n}^{\tilde{R}} - X_{v^{\tilde{R}}-1:n}^{\tilde{R}} \leq x \right\} = 1 - e^{-\lambda \sum_{i=1}^n \gamma_i x}$$

and

$$P \left\{ X_{\eta^{\tilde{R}:n}}^{\tilde{R}} - X_{\eta^{\tilde{R}}-1:n}^{\tilde{R}} \leq x \right\} = \prod_{i=1}^n (1 - e^{-\lambda \gamma_i x}).$$

3. Exceedances

Exceedances are closely related to the theory of tolerance limits first introduced by Shewart (1931) and to the invariant confidence intervals containing the future observations introduced by Bairamov and Petunin (1990). Early discussions for exceedance statistics are expressed in the works of Gumbel and von Schelling (1950), Epstein (1954), and Sarkadi (1957). Recent discussion are due to Bairamov (1997), Wesolowski and Ahsanullah (1998), Bairamov and Eryilmaz (2000), Bairamov and Kotz (2001), and Eryilmaz (2002).

Let X_1, X_2, \dots, X_k be a random sample with continuous d.f. F . Denote the order statistics of this sample by $X_{1:k} \leq X_{2:k} \leq \dots \leq X_{k:k}$. Let Y_1, Y_2, \dots, Y_m be another random sample with continuous d.f. G . Define the statistics

$$V_r^{k,m} = \#\{i \leq m : Y_i \leq X_{r:k}\}$$

and

$$W_{r,s}^{k,m} = \#\{i \leq m : Y_i \in (X_{r:k}, X_{s:k})\}.$$

The random variable $V_r^{k,m}$ shows the number of Y 's below the random threshold $X_{r:k}$. These type of statistics are used in life testing, nonparametric tests for slippage and modeling nature events such as floods and droughts.

Now, let $X_{1:n}^{\tilde{R}} \leq X_{2:n}^{\tilde{R}} \leq \dots \leq X_{n:n}^{\tilde{R}}$ be the progressive type II censored order statistics based on an absolutely continuous d.f. F . Suppose that Y_1, Y_2, \dots, Y_m are independent and identically distributed random variables with absolutely continuous d.f. G . The random variables Y_1, Y_2, \dots, Y_m can be viewed as the life times of m units which belong to the second system and independent of the first system. Let M_1 be the number of Y -failures before $X_{1:n}^{\tilde{R}}$ and M_i the number of Y -failures between $X_{i-1:n}^{\tilde{R}}$ and $X_{i:n}^{\tilde{R}}$, $i = 2, 3, \dots, n$. Recently, Ng and Balakrishnan (2004) proposed the weighted precedence test, based on M_1, M_2, \dots, M_n , for testing hypothesis $H_0 : F = G$ against $H_1 : F > G$. The joint distribution of M_1, M_2, \dots, M_n under hypothesis $H_0 : F = G$ and under Lehmann alternative can be found in Ng and Balakrishnan (2004).

It is of interest to consider the exceedance statistics

$$V_{r,m}^{\tilde{R}} = \#\left\{i \leq m : Y_i \leq X_{r:n}^{\tilde{R}}\right\}$$

which shows the number of Y 's below the random threshold $X_{r:n}^{\tilde{R}}$. This statistic can be used to compare the reliability structure of systems. Under the hypothesis $H_0 : F = G$, the exact distribution of $V_{r,m}^{\tilde{R}}$ is

$$P \left\{ V_{r,m}^{\tilde{R}} = j \right\} = \binom{m}{j} c_{r-1} \sum_{i=1}^r a_{i,r} B(j+1, m-j+\gamma_i), \quad j = 0, 1, \dots, m,$$

where $B(a, b)$ is a Beta function.

In the case of ordinary order statistics, i.e. $\tilde{R} = (0, \dots, 0)$ the statistic $V_{r,m}^{\tilde{R}}$ coincides with $V_r^{k,m}$ and its probability mass function is given by

$$P \left\{ V_r^{k,m} = j \right\} = \frac{\binom{r+j-1}{r-1} \binom{m+n-j-r}{n-r}}{\binom{m+n}{n}} = \binom{m}{j} \frac{B(j+r, m-j+n-r+1)}{B(r, n-r+1)}$$

which is the negative hypergeometric distribution. Under the hypothesis $H_0 : F = G$, it is also true that

$$P \left\{ V_{r,m}^{\tilde{R}} = j \right\} = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} P \left\{ V_1^{\gamma_i,m} = j \right\}, \quad j = 0, 1, \dots, m,$$

where the random variable $V_1^{\gamma_i,m}$ denotes the number of Y 's below the random threshold $X_{1:\gamma_i}$. $X_{1:\gamma_i}$ denotes the smallest ordinary order statistics in a sample of size γ_i .

Let us consider the statistic

$$W_{r,s,m}^{\tilde{R}} = \# \left\{ i \leq m : Y_i \in \left(X_{r:n}^{\tilde{R}}, X_{s:n}^{\tilde{R}} \right) \right\}$$

which can be represented $W_{r,s,m}^{\tilde{R}} = V_{s,m}^{\tilde{R}} - V_{r,m}^{\tilde{R}}$.

In the following we give the asymptotic distributions of $V_{r,m}^{\tilde{R}}$ and $W_{r,s,m}^{\tilde{R}}$. Proofs are omitted since they follow Bairamov (1997). See also Bairamov and Eryılmaz (2000).

Theorem 3.1. *It is true that, for $m \rightarrow \infty$*

$$\frac{V_{r,m}^{\tilde{R}}}{m} \xrightarrow{d} G(X_{r:n}^{\tilde{R}})$$

and

$$\frac{W_{r,s,m}^{\tilde{R}}}{m} \xrightarrow{d} G(X_{s:n}^{\tilde{R}}) - G(X_{r:n}^{\tilde{R}}).$$

Corollary 3.1. *Under the hypothesis $H_0 : F = G$, the asymptotic distribution of $\frac{V_{r,m}^{\tilde{R}}}{m}$ for large m is*

$$P \left\{ F(X_{r:n}^{\tilde{R}}) \leq x \right\} = P \left\{ U_{r:n}^{\tilde{R}} \leq x \right\} = 1 - c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} (1-x)^{\gamma_i}, \quad 0 < x < 1.$$

We denote $U_{r:n}^{\tilde{R}}$ as the r th progressive type II censored order statistics based on a standard uniform distribution.

4. Concomitants

Let $(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$ be a random sample from a continuous bivariate d.f. $F(x, y)$. If we denote by $X_{r:n}$ the r th order statistics of the X sample values then the Y values associated with $X_{r:n}$ is called the concomitant of the r th order statistic and is denoted by $Y_{[r:n]}$. The concomitants are of interest in selection and prediction problems. For a detailed review of the concomitants see [Bhattacharya \(1984\)](#) and [David \(1993\)](#).

In the literature, there are many papers connected with concomitants. [David et al. \(1977\)](#) derived the distribution of the rank of $Y_{[r:n]}$. [David and Galambos \(1974\)](#) discussed the asymptotic properties of the rank of $Y_{[r:n]}$. [Bairamov and Bekçi \(1999\)](#) considered the concomitants for bivariate Farlie–Gumbel–Morgenstern (FGM) distributions and they gave the recurrence relation between moments of concomitants. Recently, [Eryilmaz and Bairamov \(2003\)](#) derived the distribution of the new sample rank of an order statistics and its concomitant.

Concomitants can also be defined in the case of generalized order statistics, or progressive type II censoring. In [Kamps \(1995\)](#) the distribution of concomitant is given for Pfeifer’s record model. This way the p.d.f. of r th progressive type II censored concomitant is given by

$$f^{Y_{[r:n]}^{\tilde{R}}}(y) = \int_{-\infty}^{\infty} f^{X_{r:n}^{\tilde{R}}}(x) f(y | x) dx, \tag{8}$$

where $f(y | x)$ is the conditional p.d.f. of Y given $X = x$ and $f^{X_{r:n}^{\tilde{R}}}(x)$ is the p.d.f. of the r th progressive type II censored order statistics.

In this section, we consider the concomitants for the model of progressive type II censoring.

Lemma 4.1. *The p.d.f. of $Y_{[r:n]}^{\tilde{R}}$ can be represented as follows:*

$$f^{Y_{[r:n]}^{\tilde{R}}}(y) = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} f_{Y_{[1:\gamma_i]}}(y),$$

where $f_{Y_{[1:\gamma_i]}}(y)$ is the p.d.f. of the first concomitant of ordinary order statistics from a sample of size γ_i .

Proof. By using (8)

$$\begin{aligned} f^{Y_{[r:n]}^{\tilde{R}}}(y) &= \int_{-\infty}^{\infty} c_{r-1} f(x) \sum_{i=1}^r a_{i,r} (1 - F(x))^{\gamma_i - 1} f(y | x) dx \\ &= c_{r-1} \sum_{i=1}^r a_{i,r} \int_{-\infty}^{\infty} f(x) (1 - F(x))^{\gamma_i - 1} f(y | x) dx \\ &= c_{r-1} \sum_{i=1}^r a_{i,r} \int_{-\infty}^{\infty} \frac{1}{\gamma_i} f_{1:\gamma_i}(x) f(y | x) dx \end{aligned}$$

since $f_{Y_{[1:\gamma_i]}}(y) = \int_{-\infty}^{\infty} f_{1:\gamma_i}(x) f(y | x) dx$ the lemma is proved. \square

According to Lemma 4.1 the p.d.f. of r th concomitant of progressive type II censored order statistics is represented as the weighted sum of the first concomitant of ordinary order statistics.

By using Lemma 4.1, it is possible to represent the k th moment of $Y_{[r:n]}^{\tilde{R}}$ as follows:

$$\mu_{r,\tilde{R}}^{(k)} = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} \mu_{\gamma_i}^{(k)},$$

where $\mu_{\gamma_i}^{(k)} = EY_{[1:\gamma_i]}^k$.

For illustration, let us consider the bivariate FGM distribution with uniform marginals whose two dimensional d.f. and p.d.f. are given, respectively, by

$$\begin{aligned} F(x, y) &= xy\{1 + \alpha(1 - x)(1 - y)\}, & f(x, y) &= 1 + \alpha(1 - 2x)(1 - 2y), \\ &0 < x, y < 1, & -1 \leq \alpha \leq 1. \end{aligned} \tag{9}$$

In this case, it is easy to compute that

$$f_{Y_{[1:\gamma_i]}}(y) = 1 + \alpha(1 - 2y) \left(\frac{\gamma_i - 1}{\gamma_i + 1} \right).$$

Thus the p.d.f. of $Y_{[r:n]}^{\tilde{R}}$ is

$$f_{Y_{[r:n]}^{\tilde{R}}}(y) = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} \left\{ 1 + \alpha(1 - 2y) \left(\frac{\gamma_i - 1}{\gamma_i + 1} \right) \right\}.$$

In the case of FGM distribution the k th moment of $Y_{[1:\gamma_i]}$ is

$$\mu_{\gamma_i}^{(k)} = \frac{1}{k + 1} \left\{ 1 - \alpha \left[\frac{\gamma_i - 1}{\gamma_i + 1} \right] \frac{k}{k + 2} \right\}$$

and hence

$$\mu_{r,\tilde{R}}^{(k)} = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} \frac{1}{k + 1} \left\{ 1 - \alpha \left[\frac{\gamma_i - 1}{\gamma_i + 1} \right] \frac{k}{k + 2} \right\}.$$

Lemma 4.2. *It is true that,*

$$f_{Y_{[r:n]}^{\tilde{R}}}(y) - f_{Y_{[r-1:n]}^{\tilde{R}}}(y) = c_{r-2} \sum_{i=1}^r a_{i,r} f_{Y_{[1:\gamma_i]}}(y).$$

Proof. By using Lemma 4.1,

$$f_{Y_{[r:n]}^{\tilde{R}}}(y) - f_{Y_{[r-1:n]}^{\tilde{R}}}(y) = c_{r-1} \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} f_{Y_{[1:\gamma_i]}}(y) - c_{r-2} \sum_{i=1}^{r-1} \frac{a_{i,r-1}}{\gamma_i} f_{Y_{[1:\gamma_i]}}(y).$$

Since, $a_{i,r-1} = a_{i,r}(\gamma_r - \gamma_i)$

$$\begin{aligned} f^{Y_{[r:n]}^{\tilde{R}}}(y) - f^{Y_{[r-1:n]}^{\tilde{R}}}(y) &= c_{r-2}\gamma_r \sum_{i=1}^r \frac{a_{i,r}}{\gamma_i} f_{Y_{[1:\gamma_i]}}(y) \\ &\quad - c_{r-2} \sum_{i=1}^{r-1} \frac{a_{i,r}(\gamma_r - \gamma_i)}{\gamma_i} f_{Y_{[1:\gamma_i]}}(y) \\ &= c_{r-2} \sum_{i=1}^r a_{i,r} f_{Y_{[1:\gamma_i]}}(y). \end{aligned}$$

By using a relation which is given in Lemma 4.2, recurrence relation between $\mu_{r,\tilde{R}}^{(k)}$ and $\mu_{r-1,\tilde{R}}^{(k)}$ is given by

$$\mu_{r,\tilde{R}}^{(k)} - \mu_{r-1,\tilde{R}}^{(k)} = c_{r-2} \sum_{i=1}^r a_{i,r} \mu_{\gamma_i}^{(k)}.$$

For FGM distribution given in (9)

$$\mu_{r,\tilde{R}}^{(k)} - \mu_{r-1,\tilde{R}}^{(k)} = c_{r-2} \sum_{i=1}^r a_{i,r} \frac{1}{k+1} \left\{ 1 - \alpha \left[\frac{\gamma_i - 1}{\gamma_i + 1} \right] \frac{k}{k+2} \right\}.$$

Acknowledgements

The authors are grateful to the referees and the associate editor for their valuable comments which resulted in an improved paper.

References

Aggarwala, R., Balakrishnan, N., 1996. Recurrence relations for single and product moments of progressive type II right censored order statistics from exponential and truncated exponential distributions. *Ann. Inst. Statist. Math.* 48, 757–771.

Balakrishnan, N., Sandhu, R.A., 1996. Best linear unbiased and maximum likelihood estimation for exponential distributions under general progressive type II censored samples. *Sankhya Ser. B* 58, 1–9.

Balakrishnan, N., Cramer, E., Kamps, U., 2001a. Bounds for means and variances of progressive type II censored order statistics. *Statist. Probab. Lett.* 54, 301–315.

Balakrishnan, N., Cramer, E., Kamps, U., Schenk, N., 2001b. Progressive type II censored order statistics from exponential distributions. *Statistics* 35 (4), 537–556.

Bairamov, I.G., 1997. Some distribution free properties of statistics based on record values and characterizations of the distributions through a record. *J. Appl. Statist. Sci.* 5 (1), 17–25.

Bairamov, I.G., Bekçi, M., 1999. Concomitant of order statistics in FGM type Bivariate Uniform Distributions. *Istatistik, J. Turkish Statist. Assoc.* 1 (2), 135–144.

Bairamov, I.G., Eryilmaz, S., 2000. Distributional properties of statistics based on minimal spacing and record exceedance statistics. *J. Statist. Plann. Inference* 90, 21–33.

Bairamov, I.G., Kotz, S., 2001. On distributions of exceedances associated with order statistics and record values for arbitrary distributions. *Statist. Papers* 42 (2), 171–185.

- Bairamov, I.G., Petunin, Yu.I., 1990. Structure of invariant confidence intervals containing the main distributed mass. *Theory Probab. Appl.* 35 (1), 15–26.
- Bhattacharya, P.K., 1984. Induced order statistics. Theory and application. In: Krishnaiah, P.R., Sen, P.K. (Eds.), *Handbook of Statistics*, vol. 4. North-Holland, Amsterdam, pp. 383–403.
- David, H.A., 1993. Concomitant of order statistics: review and recent developments. In: Hoppe, F.M. (Ed.), *Multiple Comparisons, Selection and Application in Biometry*. Dekker, New York, pp. 507–518.
- David, H.A., Galambos, J., 1974. The asymptotic theory of concomitants of order statistics. *J. Appl. Probab.* 11, 762–770.
- David, H.A., O'Connell, M.J., Yang, S.S., 1977. Distribution and expected value of the rank of a concomitant of an order statistic. *Ann. Statist.* 5, 216–223.
- Epstein, B., 1954. Tables for the distribution of the number of exceedances. *Ann. Math. Statist.* 25, 762–768.
- Eryilmaz, S., 2002. Random threshold models based on multivariate observations. *J. Statist. Plann. Inference* 113, 557–568.
- Eryilmaz, S., Bairamov, I., 2003. On a new sample rank of an order statistics and its concomitant. *Statist. Probab. Lett.* 63, 123–131.
- Gumbel, E.J., von Schelling, H., 1950. The distribution of the number of exceedances. *Ann. Math. Statist.* 21, 247–262.
- Kamps, U., 1995. *A Concept of Generalized Order Statistics*. Teubner, Stuttgart.
- Kamps, U., Cramer, E., 2001. On distributions of generalized order statistics. *Statistics* 35 (3), 269–281.
- Ng, H.K.T., Balakrishnan, N., 2004. Weighted precedence and maximal precedence tests and an extension to progressive censoring. *J. Statist. Plann. Inference*, to appear.
- Sarkadi, K., 1957. On the distribution of the number of exceedances. *Ann. Math. Statist.* 28, 1021–1022.
- Shewart, W.A., 1931. *Economic Control of Quantity of Manufactured Product*. D. Van Nostrand, Princeton, NJ.
- Viveros, R., Balakrishnan, N., 1994. Interval estimation of parameters of life from progressively censored data. *Technometrics* 36, 84–91.
- Wesolowski, J., Ahsanullah, M., 1998. Distributional properties of exceedance statistics. *Ann. Inst. Statist. Math.* 50, 543–565.