# On a new family of positive quadrant dependent bivariate distributions

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**Abstract**

Recently Lai and Xie (Statistics and Probability Letters, 46 (2000) 359-364) using the uniform representation of the Farlie-Gumbel-Morgenstern bivariate distribution introduce and study continuous bivariate distributions possessing positive quadrant dependence (PQD) property with the association parameter contained between 0 and 1. We show that the association parameter in this class allowing the PQD property has a much wide range. Thus the Lai-Xie distributions may be applicable in a greater variety of situations.

## 1. Introduction

In recent years a substantial attention in the literature has been devoted to dependence prorerties among random variables (see, e.g. Joe (1997)) . Recently Lai and Xie (2000) describe a new family of positive quadrant dependent bivariate distributions. The purpose of this note is to provide an additional insight on the structure of positive dependent bivariate copulas described in Lai and Xie (2000). Recall that random variables $X$ and $Y$ are called to be positively quadrant dependent (PQD) if the inequality $P\{X \leq x, Y \leq y\} \geq P\{X \leq x\} P\{Y \leq y\}$ for all $x$ and $y$, holds. Let $F(x, y)$ denote the distribution function of $(X, Y)$ having continuous marginal cdfs $F_X(x)$ and $F_Y(y)$ with marginal pdfs $f_X = F'_X$ and $f_y = F'_Y$. As pointed out by Lai and Xie (2000) for PQD bivariate distributions the joint distribution function may be written in the form

$$F(x, y) = F_X(x)F_Y(y) + w(x, y) \text{ for all } x \text{ and } y \tag{1}$$

with nonnegative $w(x,y)$ satisfying certain regularity conditions ensuring that $F(x,y)$ is a distribution function. The so-called (bivariate) Farlie-Gumbel-Morgenstern (FGM) class of distributions originally introduced by Morgenstern (1956) for Cauchy marginal is an *important and efficient in applications* class of multivariate distributions with given marginals. This structure investigated by Gumbel (1960) for exponential marginals and further generalized by Farlie (1960). Johnson and Kotz (1975), (1977) studied the multivariate case and provided a detailed analysis of probabilistic and statistical characteristics. Huang and Kotz (1984) extended the bivariate FGM distribution in their attempts to increase the dependence between the underlying variables by introducing an additional parameter.

Let $(X,Y)$ be a bivariate absolutely continuous random variable with distribution function

$$C_\alpha(x,y) = F(x)G(y)\left\{1+\alpha A(F(x))B(G(y))\right\},$$

where $A(x) \to 0$ and $B(x) \to 0$ as $x \to 1$ and the "kernels" $A(x), B(x)$ satisfy certain regularity conditions ensuring that $C$ is a distribution function with absolutely continuous marginals $F(x)$ and $G(x)$. The admissible range of parameter $\alpha$ depends on the function $A$ and $B$. If $A(x) = B(x) = (1-x),\ \ 0 \le x \le 1$, we obtain the classical one parameter FGM family of distributions. If the marginals are uniform then for $A(x) = (1-x)^p,\ B(x) = (1-x)^p,\ p > 1$ or $A(x) = 1-x^p\ ,\ B(x) = 1-x^p,\ p > 1$ we deal with the distributions $H_\alpha(x,y) = xy(1+\alpha(1-x^p)(1-y^p), p > 1$ and $H^1_\alpha(x,y) = xy(1+\alpha(1-x)^p(1-y)^p,$ $p > 1$ which has been investigated by Huang and Kotz (1999). The admissible range of $\alpha$ for $H_\alpha(x,y)$ is

$$-\left(\max\{1,p\}\right)^{-2} \le \alpha \le p^{-1} \tag{2}$$

and for $H^1_\alpha(x,y)$ it is

$$-1 \le \alpha \le \left(\frac{p+1}{p-1}\right)^{p-1}. \tag{3}$$

While for the classical FGM the correlation between components does not exceed 1/3, the modified version allows correlation up to 0.39. Bairamov et. al. (2001) and Bairamov et. al. (2001) have considered a generalization of FGM and Sarmanov class of distributions allowing high corelation. Recently Bairamov and Kotz (2002) consider the following bivariate cdf

$$F_{n,p,\alpha}(x,y) = xy\left\{1+\alpha(1-x^n)^p(1-y^n)^p\right\}, \quad p > 1,\ \ 0 < x,y < 1,\ \ n \ge 1.$$

with the joint pdf

$$f_{n,p,\alpha}(x,y) = 1+\alpha(1-x^n)^{p-1}(1-y^n)^{p-1}\left[1-x^n(1+np)\right]\left[1-y^n(1+np)\right],$$

$$p > 1, 0 \le x,y \le 1, n \ge 1.$$

which generalizes the Huang-Kotz cases.

Here the admissible range for $\alpha$ is:

$$-\min\left\{\frac{1}{n^2}\left(\frac{1+np}{n(p-1)}\right)^{2(p-1)},1\right\}\leq\alpha\leq\frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}. \tag{3a}$$

It is of interest to note that for $F_{n,p,\alpha}(x,y)$ the correlation coefficient is equal to

$$\rho\equiv\rho(X,Y)=12\alpha t^2(p,n)$$

where $t(x,y)\equiv\frac{\Gamma(x+1)\Gamma(2/y)}{y\Gamma(x+1+2/y)}$. The so called Schweizer-Wolff index of dependence defined as

$$\sigma(X,Y)=12\int_0^1\int_0^1|F(x,y)-xy|\,dxdy$$

for uniform marginals (see e.g. Schweizer (1991)) is $\sigma(X,Y)=12\,|\alpha|\,t^2(p,n)$, which coincides up to the sign for these distributions with the correlation coefficient. The range for correlation coefficient is thus

$$-12t^2(p,n)\min\left\{\frac{1}{n^2}\left(\frac{1+np}{n(p-1)}\right)^{2(p-1)},1\right\}\leq\rho\leq12t^2(p,n)\frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}$$

The strongest positive correlation $\rho_{\max}(p)=0.4882$ is attained for $p=1.333$. The largest negative correlation is $\rho_{\min}(p)=-0.479$ , attained for $p=1.496$. For $n=3$ the strongest positive correlation is $\rho_{\max}(p)=0.5021$ attained at $p=1.506$. The fact that $\rho_{\max}(p)$ exceeds $1/2$ is gratifying for applications.

## 2. On a generalized Farlie -Gumbel-Morgenstern copulas

A bivariate copula $C(u,v)$ is a bivariate cdf with uniform marginals. Following to Lai and Xie (2000) we denote the marginal random variables of a copula by $U$ and $V$. For a joint distribution function $F(x,y)$, corresponding copula is given by

$$C(u,v)=F(F_X^{-1}(u),F_Y^{-1}(v)),$$

where $F_X^{-1}(u)=\inf\{x:F_X(x)\geq u\}$ , i.e it is the right continuous inverse function of $F_X$. For more details on copulas see, e.g. Nelsen (1999).

In Section 3.1 Lai and Xie (2000) consider the modified FGM copula of the form

$$C(u,v)=uv\left\{1+\alpha(1-u)^p(1-v)^p\right\},\ p\geq1,0\leq\alpha\leq1$$

which can also be written in the form

$$C(u,v)=uv+w(u,v), \tag{4}$$

where $w(u,v)=\alpha uv(1-u)^p(1-v)^p$, and show that $w(u,v)$ satisfy regularity conditions ensuring that $C$ is a copula. Since $w(u,v)\geq0$ for $0\leq\alpha\leq1$ the copula given in (4) is PQD.

Noting (3) one observes that if the range of $\alpha$ is extended to $0 \leq \alpha \leq \left(\frac{p+1}{p-1}\right)^{p-1}, p \geq 1$ then (4) should be PQD as well since $w(u,v) \geq 0$ for $0 \leq \alpha \leq \left(\frac{p+1}{p-1}\right)^{p-1}$. For example letting $p = 2$ we have $\left(\frac{p+1}{p-1}\right)^{p-1} = 3$. Hence (4) is a PQD copula for a wider range of $\alpha$ than $0 \leq \alpha \leq 1$.

Utilizing (2) we construct another PQD copula

$$C(u,v) = uv(1 + \alpha(1-u^p)(1-v^p), \quad p > 1, \quad 0 \leq \alpha \leq p^{-1}$$

and by considering (3a) the PQD copula

$$C(u,v) = uv\left\{1 + \alpha(1-u^n)^p(1-v^n)^p\right\}, \quad p > 1,\, n \geq 1$$

with

$$0 \leq \alpha \leq \frac{1}{n}\left(\frac{1+np}{n(p-1)}\right)^{p-1}.$$

Lai and Xie (2000) consider a bivariate function

$$C(u,v) = uv + w(u,v) = uv + \alpha u^b v^b (1-u)^a (1-v)^a, \; a, b \geq 1, \tag{5}$$

and prove that (5) is bivariate PQD copula for $0 \leq \alpha \leq 1$.

We shall show that (5) is a bivariate distribution function for $\alpha$ satisfying

$$-\min\left\{\frac{1}{[B^+(a,b)]^2}, \frac{1}{[B^-(a,b)]^2}\right\} \leq \alpha \leq -\frac{1}{B^+(a,b)B^-(a,b)}, \tag{6}$$

and possessing the PQD property for $\alpha$ satisfying

$$0 \leq \alpha \leq -\frac{1}{B^+(a,b)B^-(a,b)}, \tag{6a}$$

where

$$B^+(a,b) = \left[\frac{b(a+b-1) + \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}\right]^{b-1} \times$$
$$\times \left[1 - \frac{b(a+b-1) + \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}\right]^{a-1} \left[\frac{b(a+b-1) + \sqrt{ab(a+b-1)}}{(a+b-1)} - b\right] \tag{7}$$

and

$$B^-(a,b) = \left[\frac{b(a+b-1) - \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}\right]^{b-1} \times$$
$$\times \left[1 - \frac{b(a+b-1) - \sqrt{ab(a+b-1)}}{(a+b)(a+b-1)}\right]^{a-1} \left[\frac{b(a+b-1) - \sqrt{ab(a+b-1)}}{(a+b-1)} - b\right]. \tag{8}$$

For example letting $a = 5$ and $b = 5$ we have from (7) and (8) $-\frac{1}{B^+(a,b)B^-(a,b)} = 6.053 \times 10^4$ which is by far larger than 1.

**Theorem 1.**

$$C(u,v) = uv + w(u,v) = uv + \alpha u^b v^b (1-u)^a (1-v)^a, \ a, b \geq 1, \ 0 < u, v < 1, \quad (9)$$

with $\alpha$ satisfying (6) is the cdf of a bivariate uniform distribution. For the values of $\alpha$ satisfying (6a) $C(u,v)$ possesses the PQD property.

**Proof**. The joint pdf of (9) is

$$c(u,v) = 1 + \alpha u^{b-1}(1-u)^{a-1}\left[b-(a+b)u\right] v^{b-1}(1-v)^{a-1}\left[b-(a+b)v\right]. \quad (10)$$

We shall investigate the admissible range of $\alpha$ for distribution with the cdf (9) and pdf (10). The overall constraint on $\alpha$ is given by

$$\alpha u^{b-1}(1-u)^{a-1}\left[b-(a+b)u\right] v^{b-1}(1-v)^{a-1}\left[b-(a+b)v\right] \geq -1.$$

It is clear that $c(u,v) = 1$ on the lines $u = v = \frac{b}{a+b}$. Denote $r(u) = u^{b-1}(1-u)^{a-1}\left[b-(a+b)u\right]$. For clarity we describe the quadrants $Q1, Q2, Q3, Q4$ as follows:

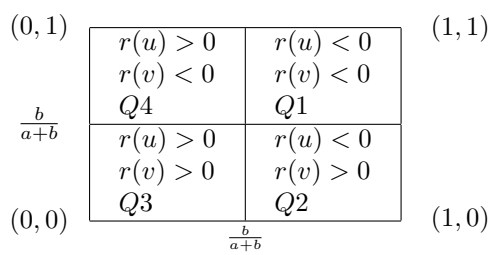


**Figure 1.** Shematic representation of the quadrants

Consider the function $r(u) = u^{b-1}(1-u)^{a-1}\left[b-(a+b)u\right]$. It is easy to verify that the solutions of

$$r'(u) = b(b-1) - 2u\left[b(a+b-1)\right] + (a+b)(a+b-1)u^2 = 0,$$

i.e the extreme points of $r(u)$ in $0 < u < 1$ are $u_* = \frac{b(a+b-1)-\sqrt{ab(a+b-1)}}{(a+b)(a+b-1)} < \frac{b}{a+b}$ and $u^* = \frac{b(a+b-1)+\sqrt{ab(a+b-1)}}{(a+b)(a+b-1)} > \frac{b}{a+b}$. Further analysis show that

$r''(u_*) < 0$ and $r''(u^*) > 0$. Thus at the point $u^*$ the function $-r(u) = u^{b-1}(1-u)^{a-1}\left[(a+b)u-b\right]$ attains its maximum and the function $r(u)$ attains its maximum at the point $u_*$.

1. In $Q1: \frac{b}{a+b} < u, v < 1$. We are required to have

$$\alpha \geq -\frac{1}{u^{b-1}(1-u)^{a-1}\left[(a+b)u-b\right]v^{b-1}(1-v)^{a-1}\left[(a+b)v-b\right]}. \tag{11}$$

Using the above arguments we obtain that in $Q1$

$$\alpha \geq -\frac{1}{\left[r(u^*)\right]^2} = -\frac{1}{\left[B^+(a,b)\right]^2}.$$

2. In $Q2: 0 < v < \frac{b}{a+b} < u < 1$.

We are required that

$$1 - \alpha u^{b-1}(1-u)^{a-1}\left[(a+b)u-b\right]v^{b-1}(1-v)^{a-1}\left[b-(a+b)v\right] \geq 0$$

or

$$\alpha \leq \frac{1}{u^{b-1}(1-u)^{a-1}\left[(a+b)u-b\right]v^{b-1}(1-v)^{a-1}\left[b-(a+b)v\right]}.$$

Therefore using the analysis above one observes that

$$\alpha \leq -\frac{1}{B^+(a,b)B^-(a,b)}.$$

3. In $Q4: 0 < u < \frac{b}{a+b} < v < 1$. By the analogy with $Q2$

$$\alpha \leq -\frac{1}{B^-(a,b)B^+(a,b)}.$$

4. In $Q3: 0 < u, v < \frac{b}{a+b}$. Here we are required to have

$$\alpha \geq -\frac{1}{u^{b-1}(1-u)^{a-1}\left[b-(a+b)u\right]v^{b-1}(1-v)^{a-1}\left[b-(a+b)v\right]}$$

The maximum attained at the point $u_*$ and hence one has

$$\alpha \geq -\frac{1}{\left[B^-(a,b)\right]^2}.$$

Thus the admissible range of $\alpha$ for which $C(u,v)$ is a copula is determined by (6) and the admissible range of $\alpha$ for which $C(u,v)$ is PQD is determined by (7).

The theorem thus proved.

We note that while Lai and Xie (2000) study the cdf

$$F(x,y) = xy + \alpha x^b y^b (1-x)^a (1-y)^a,$$

Bairamov and Kotz (1999) investigate a slightly more case with the c.d.f

$$F(x,y) = x^b y^b + \alpha x^b y^b (1-x^c)^a (1-y^c)^a.$$

The results proved in this note are not related directly to the Bairamov and Kotz (1999) generalization and deal exclusively with the Lai-Xie (2000) distribution.

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