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A NOTE ON THE RESIDUAL LIFETIMES IN A LIFE-TEST UNDER PROGRESSIVE TYPE-II RIGHT CENSORING SCHEME

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Abstract. Suppose that n independent and identically distributed items have been placed in a life-test with Progressive Type-II censoring scheme $(R_1, R_2, ..., R_m)$. In this paper, we investigate some characterizations and ordering results based on the residual lifetimes of the remaining items following the k^{th} failure in the test.

Keywords: Mean Residual Life, Generalized Pareto Distribution, Residual Life Length, Characterization, Stochastic Ordering.

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1. Introduction

In reliability analysis, the classical theory of $(n-k+1)$-out-of-n systems assumes that the n lifetimes $X_1, X_2, ..., X_n$ of components of the system are independent and identically distributed (i.i.d.) random variables, with common absolutely continuous cumulative distribution function (cdf) F, and corresponding probability density function (pdf) f. Let $X_{1:n}, X_{2:n}, ..., X_{n:n}$ be corresponding order statistics showing the failure times in the system. Recently, in [5] the authors have studied the joint distribution of residual life lengths of the remaining components after r^{th} failure $(1 \le r \le k)$ in an $(n-k+1)$-out-of-n system. After an $(n-k+1)$-out-of-n system fails, viz., after the k^{th} failure has been observed in the system, it is reasonable to break down the system and rescue the unfailed components for possible future use in other systems. In [5] it is shown that the residual life lengths $X_1^{(k)}, X_2^{(k)}, ..., X_{n-k}^{(k)}$ of the remaining components after the k^{th} failure in the system are exchangeable random variables with the joint survival function

$$\bar{F}_n^{(k)}(x_1, x_2, ..., x_{n-k}) = \int_{-\infty}^{\infty} \left[\prod_{j=1}^{n-k} \frac{\bar{F}(x_j + t)}{\bar{F}(t)} \right] dF_{k:n}(t),$$

where $\bar{F} = 1 - F$ is the survival function, and $F_{k:n}(t) = P\{X_{k:n} \le t\}$.

In this paper, we investigate residual life lengths of the remaining items after the k^{th} failure under Progressive Type-II right censoring scheme, which is widely used in reliability and life-testing. Some early works on progressive censoring was done in [8, 14, 24]. The Progressive Type-II right censored order statistics arouses the interest of many researchers, and the number of published papers has increased in the last few years. Some of the recently published papers are [1, 2, 4, 6, 17, 19, 20, 25] among many others. This subject continues to arouse the interest of many researchers, and the number of published papers in statistical literature has increased in the last few years. We also refer [9, 11, 13, 23] as the comprehensive sources in reliability theory, risk analysis, and performance analysis of networks.

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Suppose n items are placed on life-test with the corresponding failure times $X_1, X_2, ..., X_n$. Assume that the prefixed number of failures to be observed is m and the Progressive Type-II censoring scheme is given by the vector $\tilde{R} = (R_1, R_2, ..., R_m)$ with $R_1 + R_2 + \cdots + R_m + m = n$. The first failure comes at time $X_{1:n} = \min(X_1, X_2, ..., X_n) = X^{\tilde{R}}_{1:m:n}$. After the first failure, R_1 units are randomly selected and removed from the experiment. Then observing the second failure time $X^{\tilde{R}}_{2:m:n}$, the R_2 units are randomly selected and removed from the experiment, etc. After m^{th} failure, all remaining items are removed and the experiment is terminated. The failure times $X^{\tilde{R}}_{1:m:n}, X^{\tilde{R}}_{2:m:n}, ..., X^{\tilde{R}}_{m:m:n}$ are called the Progressive Type-II right censored order statistics (pcos). For simplicity of notations, we use in this paper $X_{i:m:n}$ instead of $X^{\tilde{R}}_{i:m:n}$, $i = 1, 2, ..., m$. Assuming that $X_1, X_2, ..., X_n$ have a common absolutely continuous cdf F with pdf f, the joint pdf of the first k progressively Type-II right censored order statistics is given by, $k = 1, 2, ..., m$,

$$f_{X_{1:m:n},...,X_{k:m:n}}(x_1, ..., x_k) = c_{k-1} \left[\prod_{j=1}^{k-1} f(x_j)\{\bar{F}(x_j)\}^{R_j} \right] f(x_k)\{\bar{F}(x_k)\}^{\gamma_k - 1}, \quad 0 \le x_1 < x_2 < \cdots < x_k, \tag{1}$$

where $c_{k-1} = \prod_{j=1}^{k} \gamma_j$ with $\gamma_j = \sum_{v=j}^{m} (R_v + 1)$, $j = 1, 2, ..., m$. Note that $\gamma_1 = \sum_{v=1}^{m} (R_v + 1) = n$. Here γ_j is, in fact, the number of alive items just before the p^{th} pcos. The marginal cdf of the k^{th} pcos can be expressed as

$$F_{X^{\tilde{R}}_{k:m:n}}(x) = 1 - c_{k-1} \sum_{i=1}^{k} \frac{a_i(k)}{\gamma_i} \{\bar{F}(x)\}^{\gamma_i}, \ x \ge 0,$$

where $a_i(k) = \prod_{\substack{j=1 \\ j \ne i}}^{k} \frac{1}{\gamma_j - \gamma_i}$, $i = 1, 2, ..., k$, and the empty product $\prod_{\emptyset}$ is defined to be 1. We refer the reader to [7] and the references therein for a comprehensive discussion and inferential procedures based on progressive censoring.

Throughout the paper, for any random variable W, F_W denotes the distribution function of W.

2. The residual lifetimes of the remaining items

Let $X_1, X_2, ..., X_n$ be lifetimes of n items put under Progressive Type-II right censoring scheme $(R_1, R_2, ..., R_m)$. We assume that $X_1, X_2, ..., X_n$ are i.i.d. with absolutely continuous cdf F and pdf f. Suppose that at time $X_{k:m:n} = x$, the experiment is terminated. Then it is obvious that the residual lifetimes of the remaining items in the test at time x is free of $R_{k+1}, \ldots, R_m$. Thus, without loss of generality, we can assume that $R_{k+1} = \cdots = R_m = 0$, and consider another experiment with censoring scheme $\tilde{S} = (R_1, R_2, \ldots, R_k, 0, 0, \ldots, 0)$. Note that the lifetimes of the remaining items in the original experiment are distributed as the randomly ordered values of $X^{\tilde{S}}_{k+1:k+p:n}, \ldots, X^{\tilde{S}}_{k+p:k+p:n}$, where $p = \gamma_{k+1}$. Also note that $X^{\tilde{R}}_{k:m:n} = X^{\tilde{S}}_{k:k+p:n}$. Upon using the Markov property of pcos, we have

$$f_{X^{\tilde{S}}_{k+1:k+p:n},...,X^{\tilde{S}}_{k+p:k+p:n}|X^{\tilde{S}}_{k:k+p:n}}(x_{k+1}, \ldots, x_{k+p}|x) = c \prod_{j=k+1}^{k+p} \frac{f(x_j)}{\bar{F}(x)}, \ x < x_{k+1} < \cdots < x_{k+p},$$

where c is the normalizing constant. Therefore, given $X^{\tilde{S}}_{k:k+p:n} = x$, $X^{\tilde{S}}_{k+1:k+p:n}, \ldots, X^{\tilde{S}}_{k+p:k+p:n}$ are distributed as the order statistics from an i.i.d. sample of size p with survival function $\bar{F}(y)/\bar{F}(x)$, $t > x$, where $p = p(k) = \gamma_{k+1} = \sum_{v=k+1}^{m} (R_v + 1) = R_{k+1} + R_{k+2} + \cdots + R_m + (m - k)$

is the number of survived items after k failures. If we denote by $Z_i^{(k)}$, $i = 1, 2, \ldots, p$, the randomly ordered values of $X^{\tilde{S}}_{k+1:k+p:n}, \ldots, X^{\tilde{S}}_{k+p:k+p:n}$, then given $X^{\tilde{S}}_{k:k+p:n} = x$, these $Z_i^{(k)}$'s will be i.i.d. with survival function $\bar{F}(y)/\bar{F}(x)$, $y > x$. It follows that given $X_{k:m:n} = x$, the $Z_i^{(k)}$'s are i.i.d. with common survival function $\bar{H}_x(y) = \bar{F}(y)/\bar{F}(x)$, $y > x$. The residual lifetimes $X_1^{(k)}, X_2^{(k)}, ..., X_p^{(k)}$of the remaining p items after k failures may be then defined as

$$X_i^{(k)} = Z_i^{(k)} - X_{k:m:n},\ i = 1, 2, ..., p.$$

Let $F_{X_{k:m:n}}(t)$ denote the cdf of $X_{k:m:n}$. The joint survival function of the residual lifetimes of the remaining items can be obtained as

$$\begin{aligned}
\bar{F}_p^{(k)}(x_1, x_2, ..., x_p) &= P\{X_1^{(k)} > x_1, X_2^{(k)} > x_2, ..., X_p^{(k)} > x_p\} = \\
&= \int_0^\infty P\{X_1^{(k)} > x_1, ..., X_p^{(k)} > x_p \mid X_{k:m:n} = t\} dF_{X_{k:m:n}}(t) = \\
&= \int_0^\infty P\{Z_1^{(k)} > x_1 + t, ..., Z_p^{(k)} > x_p + t \mid X_{k:m:n} = t\} dF_{X_{k:m:n}}(t) = \\
&= \int_0^\infty \left[\prod_{j=1}^p \frac{\bar{F}(x_j + t)}{\bar{F}(t)}\right] dF_{X_{k:m:n}}(t). \qquad (2)
\end{aligned}$$

It is clear from (2) that the $X_i^{(k)}$'s have exchangeable distribution. The joint density of $X_1^{(k)}, X_2^{(k)}, ..., X_p^{(k)}$ can be written as

$$f_p^{(k)}(x_1, x_2, ..., x_p) = \int_0^\infty \left[\prod_{j=1}^p \frac{f(x_j + t)}{\bar{F}(t)}\right] f_{X_{k:m:n}}(t) dt, \qquad (3)$$

and the marginal survival function of $X_i^{(k)}$ can be expressed as

$$P\{X_i^{(k)} > x\} = \int_0^\infty \frac{\bar{F}(x + t)}{\bar{F}(t)} f_{X_{k:m:n}}(t) dt.$$

Remark 1. The residual lifetimes $X_i^{(k)}$ of the remaining items after kth failure in Progressive Type-II censored experiment is closely related to the concept of the mean residual life (MRL) function. Let X be the life length of an item with absolutely continuous survival function $\bar{F}(x)$. The MRL function is defined as $\psi_F(t) = E(X - t \mid X > t)$. It is not difficult to prove that

$$E(X_1^{(k)}) = E(\psi_F(X_{k:m:n}))$$

and

$$\operatorname{cov}\left(\frac{X_1^{(k)}}{\psi_F(X_{k:m:n})}, \psi_F(X_{k:m:n})\right) = 0.$$

3. Characterizations

Let X be a lifetime (nonnegative) random variable with cdf F and survival function $\bar{F} = 1 - F$. The random variable X is said to have the generalized Pareto distribution (GPD) with parameter vector $(c,.)$ (which will be denoted by $GPD(c,.)$), where $c \in \mathbf{R}$, if $1+cX>0$ almost surely and X^*, where

$$X^* = \begin{cases} \frac{1}{c}\log(1+cX), & \text{if } \quad c \neq 0; \\ X, & \text{if } \quad c = 0, \end{cases}$$

is exponential. This family of distributions contains three distributions; for $c = 0$, the distribution is exponential, for $c > 0$, it is Pareto with linearly decreasing (increasing) failure rate (mean residual life), and for $c < 0$, it is a re-scaled beta model which has a linearly increasing (decreasing) failure rate (mean residual life). Note that for $c < 0$, the distribution is bounded above. Applications of the GPD have been extensively investigated in the literature. It is successfully applied and widely used in a number of statistical problems related to finance, insurance, hydrological frequency analysis, and other areas.

In this section, we prove some characterization results on the GPD based on the residual lifetimes of the remaining items in a life-test.

Theorem 1. Let F be an absolutely continuous cdf and c be a real number such that $cX_1 + 1 > 0$ almost surely. If

$$\frac{X_1^{(k)}}{cX_{k:m:n}+1} \stackrel{d}{=} X_1, \tag{4}$$

then F is $GPD(c,.)$.

Proof. One can check that the condition (4) simply yields

$$\int_0^\infty \frac{\bar{F}(x(1+ct)+t)}{\bar{F}(t)} f_{X_{k:m:n}}(t)dt = \bar{F}(x),$$

for every $x \geq 0$. We can write this equation as

$$\int_0^\infty \bar{F}(x(1+ct)+t)\mu(dt) = \bar{F}(x), \tag{5}$$

with the measure μ defined as $\mu(t) = f_{X_{k:m:n}}(t)/\bar{F}(t)$. In the case where $c = 0$, this is an integrated Cauchy functional equation (see, for example, [18]), and hence F is $GPD(0,.)$. When $c \neq 0$, it follows from Theorem 2 in [3] that F is $GPD(c,.)$. This completes the proof of the theorem.

Theorem 2. Let F be an absolutely continuous cdf which is strictly increasing on $[0,\omega(F))$ where $\omega(F)$ denotes the right extremity of support of F. Assume that $\theta(t)$ is a continuous function which is positive on $[0,\omega(F))$ and $\theta(0) = 1$. If

- **(a):** $\frac{X_1^{(k)}}{\theta(X_{k:m:n})}$ and $\frac{X_2^{(k)}}{\theta(X_{k:m:n})}$ are independent, or if
- **(b):** $\frac{X_1^{(k)}}{\theta(X_{k:m:n})}$ and $X_{k:m:n}$ are independent, and for each $x > 0$, $\bar{F}(x\theta(t)+t)/\bar{F}(t)$ is a monotone function of t,

then F is $GPD(c,.)$, for some $c \in \mathbf{R}$.

Proof. Let assumption (a) hold. For any $x_1, x_2 > 0$, we can obtain the joint survival function

$$P\left\{\frac{X_1^{(k)}}{\theta(X_{k:m:n})} > x_1, \frac{X_2^{(k)}}{\theta(X_{k:m:n})} > x_2\right\} = \int_0^\infty \left[\prod_{j=1}^2 \frac{\bar{F}(x_j\theta(t)+t)}{\bar{F}(t)}\right] dF_{X_{k:m:n}}(t).$$

The independence assumption implies that

$$\int_0^\infty \left[\prod_{j=1}^{2} \frac{\bar{F}(x_j\theta(t)+t)}{\bar{F}(t)}\right] dF_{X_{k:m:n}}(t) = \int_0^\infty \frac{\bar{F}(x_1\theta(t)+t)}{\bar{F}(t)} dF_{X_{k:m:n}}(t) \times \times \int_0^\infty \frac{\bar{F}(x_2\theta(t)+t)}{\bar{F}(t)} dF_{X_{k:m:n}}(t),$$

or equivalently

$$\operatorname{cov}\left(\frac{\bar{F}(x_1\theta(X_{k:m:n})+X_{k:m:n})}{\bar{F}(X_{k:m:n})}, \frac{\bar{F}(x_2\theta(X_{k:m:n})+X_{k:m:n})}{\bar{F}(X_{k:m:n})}\right) = 0.$$

Since x_1, x_2 are arbitrary, we get, for each $x > 0$, that

$$\operatorname{var}\left(\frac{\bar{F}(x\theta(X_{k:m:n})+X_{k:m:n})}{\bar{F}(X_{k:m:n})}\right) = 0.$$

This implies that for each x, $\bar{F}(x\theta(X_{k:m:n})+X_{k:m:n})/\bar{F}(X_{k:m:n})$ is degenerate. Hence, we obtain the ratio $\bar{F}(x\theta(y)+y)/\bar{F}(y)$ to be independent of y for each $y \in (0, \omega(F))$ and $x \in \mathbf{R}_+$, say $\phi(x)$. By considering the limits as $y \to 0^+$, and using the right continuity of $\bar{F}(y)$ and the continuity of $\theta(y)$, we can conclude that $\phi(x) = \bar{F}(x)$. Now the desired result follows from [16] ; that is, F is $GPD(c,.)$ for some $c \in \mathbf{R}$. This proves the result when the assumption (a) holds.

Now let the condition (b) hold. After some manipulations, we can write the joint survival function of

$$\frac{X_1^{(k)}}{\theta(X_{k:m:n})}, \frac{X_2^{(k)}}{\theta(X_{k:m:n})}, ..., \frac{X_p^{(k)}}{\theta(X_{k:m:n})}, X_{k:m:n}$$

as follows,

$$P\left\{\frac{X_1^{(k)}}{\theta(X_{k:m:n})} > x_1, \frac{X_2^{(k)}}{\theta(X_{k:m:n})} > x_2, ..., \frac{X_p^{(k)}}{\theta(X_{k:m:n})} > x_p, X_{k:m:n} > y\right\} = = \int_y^\infty \left[\prod_{j=1}^{p} \frac{\bar{F}(x_j\theta(t)+t)}{\bar{F}(t)}\right] dF_{X_{k:m:n}}(t),$$

and therefore we have

$$P\left\{\frac{X_1^{(k)}}{\theta(X_{k:m:n})} > x, X_{k:m:n} > y\right\} = \int_y^\infty \frac{\bar{F}(x\theta(t)+t)}{\bar{F}(t)} dF_{X_{k:m:n}}(t),\ x, y \geq 0. \tag{6}$$

Thus, by the independence assumption, we get

$$\int_y^\infty \frac{\bar{F}(x\theta(t)+t)}{\bar{F}(t)} dF_{X_{k:m:n}}(t) = \int_0^\infty \frac{\bar{F}(x\theta(t)+t)}{\bar{F}(t)} dF_{X_{k:m:n}}(t) \int_y^\infty dF_{X_{k:m:n}}(t),$$

which can be written as

$$\operatorname{cov}\left(I_{\{y \leq X_{k:m:n} < \infty\}}, \frac{\bar{F}(x\theta(X_{k:m:n})+X_{k:m:n})}{\bar{F}(X_{k:m:n})}\right) = 0,$$

for any $y \in (0, \omega(F))$. Using Tchebychev's second inequality (see [5]) we conclude that for each $x > 0$, $\bar{F}(x\theta(X_{k:m:n})+X_{k:m:n})$ is degenerate. This means that F is $GPD(c, 0)$ for some $c \in \mathbf{R}$. The proof is complete.

Theorem 3. Let F be an absolutely continuous cdf, and c be a real number such that $cX_1 + 1 > 0$ almost surely. Then

$$E\left\{\phi\left(\frac{X_1^{(k)}}{cX_{k:m:n}+1}\right)\ \middle|\ X_{k:m:n}=x\right\}=\alpha,\ x\geq 0, \tag{7}$$

for some nonnegative and strictly increasing function $\phi(x)$, if and only if F is $GPD(c,.)$, where α is a positive constant.

Proof. From (6), it follows that for each $x, t > 0$,

$$P\left\{\frac{X_1^{(k)}}{cX_{k:m:n}+1}>t\ \middle|\ X_{k:m:n}=x\right\}=\frac{\bar{F}(x+(1+cx)t)}{\bar{F}(x)}.$$

Let equality in (7) hold. Then we can conclude that

$$\int_0^\infty \bar{F}(x+(1+cx)t)d\phi(t)=\alpha\bar{F}(x),$$

which is the same integral equation in (5) with $\mu = \phi/\alpha$. As we see in the proof of Theorem 1, the only solution is the survival function of $GPD(c,.)$.

4. Ordering results

Recall that F is said to be new better than used (NBU) if for every $t, x \geq 0$, we have $\bar{F}(x+t) \leq \bar{F}(x)\bar{F}(t)$, and F is said to be new worse than used (NWU) if for every $t, x \geq 0$, we have $\bar{F}(x+t) \geq \bar{F}(x)\bar{F}(t)$. The following theorem describes the properties of NBU and NWU based on stochastic comparisons between $X_1^{(k)}$ and X_1. The proof is simple and hence is omitted.

Theorem 4. If F is NBU (NWU), then $X_1^{(k)} \leq_{st} X_1$ ($X_1 \leq_{st} X_1^{(k)}$).

In the following theorems, we provide same stochastic orderings of the residual lifetimes of the remaining items in a life-test. For two random variables X and Y, with respective density functions f, and g, and survival functions $\bar{F}$ and $\bar{G}$, X is said to be smaller than Y in likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x, and, X is said to be smaller than Y in hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x. For a comprehensive discussion on various concepts of stochastic ordering, we refer the reader to [21]. We recall that a function $h(x,y)$ is said to be totally positive of order 2 (TP$_2$) if $h(x,y) \geq 0$ and

$$h(x_1,y_1)h(x_2,y_2)-h(x_1,y_2)h(x_2,y_1)\geq 0,$$

whenever $x_1 < x_2$ and $y_1 < y_2$. If the above inequality is reversed, then $h(x,y)$ is said to be reverse regular of order 2 (RR$_2$). For more details on TP$_2$ and RR$_2$ functions, see [10]. First note that, using (1), the density function of $X_{k:m:n}$ can be written as

$$f_{X_{k:m:n}}(x)=c_{k-1}f(x)\{\bar{F}(x)\}^{\gamma_k-1}\xi_k(F(x)),$$

where $\xi_1 \equiv 1$, $\xi_k(F(x)) = \int\limits_A \prod\limits_{j=1}^{k-1}(1-u_j)^{R_j}du_j,\ k=2,3,...,m$, and $A=\{(u_1,...,u_{k-1}): 0<u_1< u_2<\cdots<u_{k-1}<F(x)\}$.

Theorem 5. Let $X_1, X_2, ..., X_n$ be i.i.d. nonnegative random variables with distribution F and density f and given $X_{k:m:n} = x$, denote by $X_i^{(k)},\ i = 1,2,...,p$, the residual lifetimes of the remaining items. Also let $Y_1, Y_2, ..., Y_n$ be other i.i.d. nonnegative random variables with distribution G and density g and given $Y_{k:m:n} = x$, denote by $Y_i^{(k)},\ i = 1,2,...,p$, the residual lifetimes of the remaining items. If $X_1 \leq_{lr} Y_1$ and f and g are logconvex, then $X_1^{(k)} \leq_{lr} Y_1^{(k)}$.

Proof. We denote F by H_1, G by H_2, f by h_1, and g by h_2. Let $\bar{H}_i = 1 - H_i,\ i = 1,2$. We need to prove that

$$\begin{aligned} h^{(k)}_{i,m,n}(x) &= \int_0^\infty \frac{h_i(x+t)}{\bar{H}_i(t)} c_{k-1} h_i(t)\{\bar{H}_i(t)\}^{\gamma_k - 1}\xi_k(H_i(t))dt = \\ &= c_{k-1}\int_0^\infty h_i(x+t)h_i(t)\{\bar{H}_i(t)\}^{\gamma_k-2}\xi_k(H_i(t))dt \end{aligned}$$

is TP_2 in $(i,x) \in \{1,2\} \times \mathbf{R}_+$. First we use an inductive argument (on $k = 1,2,...,m$) to show that $\xi_k(H_i(t))$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. The proof for the case $k = 1$ is trivial, and for $k = 2$, it is easy to verify that

$$\xi_2(H_i(t)) = \frac{1 - \{\bar{H}_i(t)\}^{R_1+1}}{R_1 + 1}$$

is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. Now assume that $\xi_{k-1}(H_i(t))$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. It is easily shown that

$$\xi_k(H_i(t)) = \int_{\mathbf{R}_+} I_{[0,t]}(\omega)h_i(\omega)\{\bar{H}_i(\omega)\}^{R_k-1}\xi_{k-1}(H_i(\omega))d\omega,\ k = 2,3,...,m.$$

From the assumption $X_1 \le_{lr} Y_1$, we get that $h_i(\omega)$ and $\{\bar{H}_i(\omega)\}^{R_k-1}$ are both TP_2 in $(i,\omega) \in \{1,2\} \times \mathbf{R}_+$. Also, the indicator function $I_{[0,t]}(\omega)$ is TP_2 in $(\omega,t) \in \mathbf{R}_+ \times \mathbf{R}_+$. On noting that a product of TP_2 kernels is TP_2, we can use the Basic Composition Formula (see, for example, [10]) to get that $\xi_k(H_i(t))$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$.

From the fact that $1 \le \gamma_m < \gamma_{m-1} < \cdots < \gamma_1$, it can be easily seen that $\gamma_k \ge 2$, and hence $\{\bar{H}_i(t)\}^{\gamma_k-2}$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. The logconvexity of h_i means that $h_i(x+t)$ is TP_2 in $(x,t) \in \mathbf{R}_+ \times \mathbf{R}_+$, and the assumption $X_1 \le_{lr} Y_1$ implies that $h_i(t)$ is TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$, and $h_i(x+t)$ is TP_2 in $(i,x) \in \{1,2\} \times \mathbf{R}_+$ and in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. Using again, the fact that a product of TP_2 kernels is TP_2, and applying Theorem 5.1 on page 123 of [10], we find that $h^{(k)}_{i,m,n}(x)$ is TP_2 in $(i,x) \in \{1,2\} \times \mathbf{R}_+$; that is $X^{(k)}_1 \le_{lr} Y^{(k)}_1$.

Remark 2. According to Theorem 1.C.72 in [21], logconvexity of densities of X_1 and Y_1, together with the assumption $X_1 \le_{lr} Y_1$, implies that $X_1 \le_{lr\downarrow} Y_1$; that is X_1 is smaller than Y_1 in the down shifted likelihood ratio order. This means that $g(x+t)/f(t)$ is increasing in $t \ge 0$ for all $x \ge 0$.

Theorem 6. Let X_i, Y_i, $X^{(k)}_i$ and $Y^{(k)}_i$ be defined as in Theorem 5, and $\bar{F}$ and $\bar{G}$ denote the survival functions of X_i and Y_i, respectively. If $X_1 \le_{lr} Y_1$, and if $\bar{F}$ and $\bar{G}$ are logconvex (i.e. X_1 and Y_1 are DFR), then $X^{(k)}_1 \le_{hr} Y^{(k)}_1$.

Proof. To prove the theorem, we show that

$$\begin{aligned} \bar{H}^{(k)}_{i,m,n}(x) &= \int_0^\infty \frac{\bar{H}_i(x+t)}{\bar{H}_i(t)} h_i(t)\{\bar{H}_i(t)\}^{\gamma_k-1}\xi_k(H_i(t))dt = \\ &= \int_0^\infty \bar{H}_i(x+t)h_i(t)\{\bar{H}_i(t)\}^{\gamma_k-2}\xi_k(H_i(t))dt \end{aligned}$$

is TP_2 in $(i,x) \in \{1,2\} \times \mathbf{R}_+$. By following an argument similar to that in the proof of Theorem 5, we have

$$h_i(t)\{\bar{H}_i(t)\}^{\gamma_k-2}\xi_k(H_i(t))$$

to be TP_2 in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. The logconvexity of $\bar{H}_i$ means that $\bar{H}_i(x+t)$ is TP_2 in $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$. The assumption $X_1 \le_{lr} Y_1$ implies that $X_1 \le_{hr} Y_1$, which, in turn implies that $\bar{H}_i(x+t)$ is TP_2 both in $(i,x) \in \{1,2\} \times \mathbf{R}_+$ and in $(i,t) \in \{1,2\} \times \mathbf{R}_+$. Thus, by Theorem 5.1 on page 123 of [10], we see that $\bar{H}^{(k)}_{i,m,n}(x)$ is TP_2 in $(i,x) \in \{1,2\} \times \mathbf{R}_+$. This means that $X_1^{(k)} \leq_{hr} Y_1^{(k)}$.

Theorem 7. Let $X_1, X_2, ..., X_n$ be i.i.d. nonnegative random variables with absolutely continuous cdf F and pdf f.

(a): If f is logconvex (logconcave), then $X_1^{(k-1)} \leq_{lr} X_1^{(k)}$ ($X_1^{(k)} \leq_{lr} X_1^{(k-1)}$), $k = 2, 3, ..., n$.

(b): If F is DFR (IFR), then $X_1^{(k-1)} \leq_{hr} X_1^{(k)}$ ($X_1^{(k)} \leq_{hr} X_1^{(k-1)}$), $k = 2, 3, ..., n$.

Proof. To prove part (a), note that from (2), the density of $X_1^{(k)}$ can be obtained as

$$f_{X_1^{(k)}}(x) = \int_0^\infty \frac{f(x+t)}{\bar{F}(t)} f_{X_{k:m:n}}(t)dt.$$

It is proved in [12] that $X_{k-1:m:n} \leq_{lr} X_{k:m:n}$. This is equivalent to say that $f_{X_{k:m:n}}(t)$ is TP_2 in $(k,t) \in \{1,2,...,n\} \times \mathbf{R}_+$. The logconvexity (logconcavity) of f means that $f(x+t)$ is TP_2 in $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$. Thus, it follows from the Basic Composition Formula (see [10]) that $f_{X_1^{(k)}}(x)$ is TP_2 in $(k,x) \in \{1,2,...,n\} \times \mathbf{R}_+$; that is $X_1^{(k-1)} \leq_{lr} X_1^{(k)}$ ($X_1^{(k)} \leq_{lr} X_1^{(k-1)}$).

Part (b) can be proved similarly on noting that if F is DFR (IFR), then $\bar{F}$ is logconvex (logconcave), which in turn, implies that $\bar{F}(x+t)$ is TP_2 (RR_2) in $(t,x) \in \mathbf{R}_+ \times \mathbf{R}_+$.

Next, we prove some other properties of the residual lifetimes of the remaining items in a test, when the parent density or survival function is logconvex. As before we assume that the underlying distribution is absolutely continuous.

Theorem 8.

(a): If X_1 has a logconvex density f, then the joint density $f_p^{(k)}(x_1, x_2, ..., x_p)$ of $(X_1^{(k)}, X_2^{(k)}, ..., X_p^{(k)})$ is TP_2 in pairs.

(b): If X_1 has a logconvex survival function, then the joint survival function of $(X_1^{(k)}, X_2^{(k)}, ..., X_p^{(k)})$ is TP_2 in pairs.

Proof. First, we prove part (a). By (3), the the joint density of $(X_1^{(k)}, X_2^{(k)}, ..., X_p^{(k)})$ can be written as

$$f_p^{(k)}(x_1, x_2, ..., x_p) = \int_0^\infty \left[\prod_{j=1}^p f(x_j+t)\right] \frac{f_{X_{k:m:n}}(t)}{\bar{F}(t)} dt.$$

The logconvexity of f implies that $f(x_j+t)$ is TP_2 in $(x_j,t) \in \mathbf{R}_+ \times \mathbf{R}_+$. Therefore, $f_p^{(k)}(x_1, x_2, ..., x_p)$ is TP_2 in pairs. The proof of part (b) is omitted since it is similar to the proof of part (a) (see equation (2)).

In the literature, various notions of positive dependence of two random vectors have been introduced. "Conditionally i.i.d." is one of these concepts. In the following, we mention an interesting result concerning conditionally i.i.d. random variables.

Let $X_1, X_2, ..., X_n$ be conditionally i.i.d. (this, of course, implies that X_i's are exchangeable), $Y_1, Y_2, ..., Y_n$ are i.i.d., and all the X_i's and Y_i's have the same marginal distributions. In [22] it is shown that under these conditions

$$(F_{Y_{1:n}}(t), F_{Y_{2:n}}(t), ..., F_{Y_{n:n}}(t)) \succ (F_{X_{1:n}}(t), F_{X_{2:n}}(t), ..., F_{X_{n:n}}(t)), \ \forall t \in \mathbf{R},$$

and

$$(Eh(Y_{1:n}), Eh(Y_{2:n}), ..., Eh(Y_{n:n})) \succ (Eh(X_{1:n}), Eh(X_{2:n}), ..., Eh(X_{n:n})),$$

for all monotone functions h, such that the expectations exist. Here $\succ$ denotes the majorization order, see [15]. A vector $\mathbf{a} = (a_1, a_2, ..., a_n)$ is said to be smaller in the majorization order than the vector $\mathbf{b} = (b_1, b_2, ..., b_n)$ (denoted $\mathbf{b} \succ \mathbf{a}$) if $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and if $\sum_{i=1}^{j} a_{[i]} \leq \sum_{i=1}^{j} b_{[i]}$ for $j = 1, 2, ..., n-1$, where $a_{[i]}$ and $b_{[i]}$ are the i^{th} largest elements of $\mathbf{a}$ and $\mathbf{b}$, respectively. Let us define the exchangeable random variables

$$W_i = \frac{X_i^{(k)}}{cX_{k:m:n} + 1}, \; i = 1, 2, ..., p,$$

where c is a real valued constant, such that $cX_{k:m:n} + 1 > 0$ almost surely. The next theorem provides a multivariate ordering between X_i's and W_i's in the case where the parent distribution is GPD.

Theorem 9. Let $X_1, X_2, ..., X_n$ be i.i.d. random variables with common $GPD(c, .)$ distribution for some $c \in \mathbf{R}$. Then

$$\left(F_{X_{1:p}}(t), F_{X_{2:p}}(t), ..., F_{X_{p:p}}(t)\right) \succ \left(F_{W_{1:p}}(t), F_{W_{2:p}}(t), ..., F_{W_{p:p}}(t)\right),$$

and

$$(Eh(X_{1:p}), Eh(X_{2:p}), ..., Eh(X_{n:p})) \succ (Eh(W_{1:p}), Eh(W_{2:p}), ..., Eh(W_{p:p})),$$

for all monotone functions h such that the expectations exist.

Proof. It is known that for the GPD, $W_i \stackrel{d}{=} X_i$ (see Theorem 1) and $W_1, W_2, ..., W_n$ are conditionally i.i.d. Now the result follows easily from the result of [22].

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